METR 3113 | Quiz #2 | October 16, 2006

Instructions: NO CALCULATORS. Use the distributed BLUE BOOK for submitting your answers. Print your name, and sign your legal signature on the front of the blue book. Print your name on the back of the blue book. Everybody must answer question 1. Then answer either 2 or 3, and 4 or 5. INDICATE ON THE FRONT OF THE BLUE BOOK WHICH 3 QUESTIONS YOU WANT TO BE GRADED.

1. **REQUIRED.** (2 pts each, 20 pts total)

   (a) Neglecting friction and the Coriolis force, the one-dimensional, horizontal equation of motion in meteorology is
   $$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \ . \tag{1}$$
   Suppose p decreases with x at a rate of 20 mb per per thousand kilometers. Assume air density is 1 kg m^{-3}. The parcel starts at $t = 0$ with $u = 0$. What is u after 1000 s?

   (b) Write down the vertical equation of motion, analogous to (1). Of the three terms, what are the typical values of the two largest terms?

   (c) Given knowledge of the average sealevel pressure on Earth p_0, how could we calculate the mass of the entire atmosphere?

   (d) Given
   $$a_\theta = r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} \tag{2}$$
   and $v_r \equiv \frac{dr}{dt}$ and $v_\theta \equiv r\frac{d\theta}{dt}$, convert a_θ to be expressed in terms of v_r and v_θ.

   (e) Show that
   $$\frac{d}{dt}(rv_\theta) = 0 \tag{3}$$
   is consistent with $a_\theta = 0$.

   (f) Somewhat analogous to (1), write down the radial equation of motion for cyclostrophic balance, meaning for a parcel with speed v in a circular orbit of radius r. Using this equation, estimate $\frac{\partial p}{\partial r}$ for a vortex in the lower troposphere, at a radius of $r = 100$ m, where the azimuthal speed is $v = 100$ m s^{-1}. Give the answer in units of millibars per meter.

   (g) Suppose a cockroach walks along a spiral $r(\theta) = r_0\theta$ with $\frac{d\theta}{dt} = \omega$. The velocity vector is $\vec{v} = \frac{dr}{dt}\widehat{\mathbf{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}}$. Find v^2 as a function of r, r_0 and ω.

   (h) Let y be the elevation of a parcel of air above its equilibrium position. Suppose $y(t)$ is governed by:
   $$\frac{d^2y}{dt^2} + \omega^2 y = 0 \ . \tag{4}$$

Derive the energy equation from (4). Write down the quantity that the energy equation shows is conserved.

(i) Suppose at $t = 0$ $\frac{dy}{dt} = w_0$ and $y = 0$. Use your previous result to forecast how high the parcel will rise (forecast the maximum value of y).

(j) In (4), how is ω^2 related to the environmental potential temperature $\overline{\Theta}(z)$?

2. **Buoyancy Choice 1. 10 pts.** Suppose the vertical velocity w of a parcel of air in a thunderstorm is governed by:

$$\rho \frac{dw}{dt} = -\rho g - \frac{\partial p}{\partial z} \; . \tag{5}$$

Suppose outside the storm the density ρ varies with z as $\rho = \overline{\rho}(z)$.

(a) (1 pt) Let the pressure outside the storm be denoted as $p = \overline{p}(z)$. If the atmosphere is hydrostatic outside the storm, relate $\frac{\partial \overline{p}(z)}{\partial z}$ to $\overline{\rho}(z)$.

(b) (3 pts) Knowledge of the pressure field outside the storm can be used to simplify (5). Show how that simplification leads to a concept of acceleration by buoyancy in (5).

(c) (3 pts) Express the buoyancy term in (b) in terms of temperature T and g.

(d) (2 pts) Suppose the updraft of the thunderstorm is 3 K warmer than than outside the storm, at the same elevation. Estimate the magnitude of of the vertical acceleration (positive is upward).

(e) (1 pts) If a parcel accelerates from rest, how fast is it going after 10^3 seconds?

3. **Buoyancy Choice 2. 10 pts.** Suppose the vertical velocity w of a parcel of air in a thunderstorm is governed by:

$$\frac{dw}{dt} = b \tag{6}$$

where b is the buoyancy. Suppose $b = b(z)$, where z is elevation, and we seek information about $w(z)$ from (6).

(a) (2 pts) Show that

$$\frac{d}{dt} w(z) = \frac{d}{dz} \left(\frac{w^2}{2} \right) \tag{7}$$

(b) (4 pts) Use (6) and (7) to find how $w_2^2 - w_1^2$ is related to $b(z)$. We define $w_2 \equiv w(z_2)$ and $w_1 \equiv w(z_1)$.

(c) (4 pts) Write down how b could be expressed in term of potential temperature. Be sure to clearly define all the symbols that you write down.

4. **Polar Coordinates Choice 1. 10 pts.** Consider motion in a plane. The position vector in Cartesian coordinates can be written

$$\vec{r} = x\,\widehat{\mathbf{\imath}} + y\,\widehat{\mathbf{\jmath}}\ . \tag{8}$$

The Cartesian coordinates are related to polar coordinates by $x = r\cos\theta$ and $y = r\sin\theta$. With

$$\widehat{\mathbf{r}} \equiv \cos\theta\,\widehat{\mathbf{\imath}} + \sin\theta\,\widehat{\mathbf{\jmath}}\ , \tag{9}$$

we write

$$\vec{r} = r\,\widehat{\mathbf{r}}\ . \tag{10}$$

We also define

$$\widehat{\boldsymbol{\theta}} \equiv -\sin\theta\,\widehat{\mathbf{\imath}} + \cos\theta\,\widehat{\mathbf{\jmath}}\ . \tag{11}$$

(a) Derive the value of $\widehat{\mathbf{r}} \times \widehat{\boldsymbol{\theta}}$. (1 pt)

(b) Evaluate, and simplify, $\frac{d\widehat{\mathbf{r}}}{dt}$. (1 pt)

(c) Evaluate, and simplify, $\frac{d\widehat{\boldsymbol{\theta}}}{dt}$. (1 pt)

(d) Show that

$$\vec{v} \equiv \frac{d\vec{r}}{dt} = \frac{dr}{dt}\widehat{\mathbf{r}} + r\frac{d\theta}{dt}\widehat{\boldsymbol{\theta}}\ . \tag{12}$$

(2 pts)

(e) Show that

$$\vec{a} \equiv \frac{d^2\vec{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\widehat{\mathbf{r}} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\widehat{\boldsymbol{\theta}}\ . \tag{13}$$

(3 pts)

(f) Suppose in a tornado $a_\theta = 0$. Suppose for a particular parcel $\frac{d\theta}{dt} = 0.1$ rad s^{-1} at r=100 m. What is $\frac{d\theta}{dt}$ when the parcel reaches r=50 m? (2 pts)

5. **Polar Coordinates Choice 2. 10 pts.** Consider a vortex with a nonrotating and motionless core, for $r < r_c$. Outside the core, or $r \geq r_c$, the azimuthal velocity is:

$$v(r) = v_c\frac{r_c}{r}\ . \tag{14}$$

Assume the density ρ is a constant.

(a) Make a very rough sketch of v as a function of r. (2 pts)

(b) The vortex is in cyclostrophic balance. Write down the equation for cyclostrophic balance. In particular, write down how $\frac{dp}{dr}$ is related to density ρ, v and r. (2 pts)

(c) Solve the above equation for cyclostrophic balance. In other words, find $p(r)$, assuming ρ is a constant, for $0 < r < r_c$ and $r \geq r_c$. (2 pts)

(d) Find $p(\infty) - p(0)$ in terms of v_c and ρ. (2 pts)

(e) If $v_c = 100$ m s^{-1} and $\rho = 1$ kg m^{-3}, find $p(\infty) - p(0)$ in mb. (2 pts)