METR 3113 **Practice Problems for Preparation for Quiz 1** 2007

A list of practice problems for Quiz 1 (September 24, 2007) follows. This is the version from September 23, 2007. Hopefully equation (13) is correct this time!

1. Consider

$$\widehat{\mathbf{I}} = \sqrt{3}/2\widehat{\mathbf{\imath}} + 1/2\widehat{\mathbf{\jmath}}\ . \tag{1}$$

$$\widehat{\mathbf{J}} = -1/2\widehat{\mathbf{\imath}} + \sqrt{3}/2\widehat{\mathbf{\jmath}}\ . \tag{2}$$

   (a) Prove $\widehat{\mathbf{I}} \cdot \widehat{\mathbf{I}} = 1$, $\widehat{\mathbf{I}} \cdot \widehat{\mathbf{J}} = 0$, $\widehat{\mathbf{J}} \cdot \widehat{\mathbf{J}} = 1$, $\widehat{\mathbf{I}} \times \widehat{\mathbf{J}} = \widehat{\mathbf{k}}$ (thus $\widehat{\mathbf{k}} = \widehat{\mathbf{K}}$).

   (b) What is the angle between $\widehat{\mathbf{I}}$ and $\widehat{\mathbf{\imath}}$?

   (c) Make a sketch of these two sets of units vectors (these two bases).

   (d) Consider $\vec{a} = 2\widehat{\mathbf{\imath}} + 4\widehat{\mathbf{\jmath}}$. Suppose we wish to write $\vec{a} = a_X\widehat{\mathbf{I}} + a_Y\widehat{\mathbf{J}}$. Find $a_X$ and $a_Y$. Hint: the most efficient way to do this is $a_X = \widehat{\mathbf{I}} \cdot \vec{a}$ and $a_Y = \widehat{\mathbf{J}} \cdot \vec{a}$.

   (e) Now suppose $\vec{b} = 1\widehat{\mathbf{\imath}} + 1\widehat{\mathbf{\jmath}}$. Find $\vec{a} \cdot \vec{b}$ using both the $\widehat{\mathbf{\imath}}, \widehat{\mathbf{\jmath}}$ basis and the $\widehat{\mathbf{I}}, \widehat{\mathbf{J}}$ basis. (Both computations should give the same answer!)

   (f) What is the angle between $\vec{a}$ and $\vec{b}$?

   (g) Is $\vec{a} + \vec{b}$ *rotationally invariant*?

   (h) Is $\vec{a} \cdot \vec{b}$ *rotationally invariant*?

   (i) Is $\vec{a} \heartsuit \vec{b} \equiv a_x b_y + a_y b_z + a_z b_x$ *rotationally invariant*?

   (j) Find $\vec{a} \times \vec{b}$ using both bases. (Both computations should give the same answer!)

2. Prove $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$ by grinding out the expansion of all the terms. I suggest letting $\vec{e} \equiv \vec{b} \times \vec{c}$, find $e_x = b_y c_z - b_z c_y$, etc. Perhaps also let $\vec{f} \equiv \vec{a} \times (\vec{b} \times \vec{c})$, and find $f_x = a_y e_z - a_z e_y = a_y b_x c_y - a_y b_y c_x - a_z b_z c_x + a_z b_x c_z$, There, you are now 1/3 done with the left hand side :)

3. Prove, using known vector identities that $(\vec{a} \times \vec{b}) \cdot (\vec{c} \times \vec{d}) = (\vec{b} \cdot \vec{d})(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{d})(\vec{b} \cdot \vec{c})$

4. Using the above, prove the magnitude of a cross produce is product of the magnitudes of the input vectors, times the sine of the angle between them.

5. Consider $P \equiv \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$ and investigate how $P$ varies in the $(\theta, \phi)$ plane on a trajectory $\theta = t$ and $\phi = b - t$. Specifically, calculate $dP/dt$. Show that $dP/dt = 0$. You have thus shown that on any trajectory $P$ is a constant, and thus $P = P(b) = P(\theta + \phi)$. Next deduce that $P(\theta + \phi) = \cos(\theta + \phi)$.

6. Given $\cos(\theta+\phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$, prove by differentiation that $\sin(\theta+\phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$.

7. Prove the Law of Cosines

8. Prove the Law of Sines

9. Find the pressure $p$ as a function of height $z$ in an atmosphere of constant temperature $T$ given that

$$\frac{dp}{dz} = -\frac{g}{RT}p \tag{3}$$

where $g$ is the acceleration due to gravity and $R$ is the gas constant for air. Write the surface pressure $p(0)$ as $p_0$. Write a calculator formula to give the height of a pressure surface in kilometers as a function of pressure in millibars. Assume $T = 255$ K and a surface pressure of 1013 mb. At what height $z$ is $p = 200$ mb? (See 2004 ps1.pdf for solutions to this problem, and to the next two.)

10. Exponential adjustment towards an asymptote that is not zero is very common in nature. Here we consider a small raindrop moving through still air according to

$$\frac{dw}{dt} = -g - kw \tag{4}$$

where $w$ is the vertical velocity and $k$ is a constant particular for the size of the drop and the density of the surrounding air (not the universal $k$ you saw before). Find $w(t)$ given that $w(0) = 0$. Make a sketch of the solution.

Here is the way I solve (4). I first find the equilibrium value of $w$ for which $dw/dt = 0$. I write that value as $w_e$; here $w_e \equiv -\frac{g}{k}$. Next I write $w(t) = w_e + w_p(t)$ where $w_p$ is the *perturbation from the equilibrium value* of $w$. I then find a simpler differential equation for $w_p(t)$, solve for $w_p(t)$, and add that solution to $w_e$.

11. Invert your solution for $w(t)$ to find $kt$ as function of $w$. Plot $kt$ as a function of $\frac{w}{w_e}$. How long do you need to wait until $\frac{w}{w_e} = 0.9$? (This means find the value of $kt$. If $k$ were given, you could also easily find $t$). How long do you need to wait until $\frac{w}{w_e} = 1.1$?

12. This is from the 2004 ps1.pdf:

(a) Show that for a satellite in a circular orbit of radius $r$ around Earth (which has radius $a$) that the angular frequency $\omega$ is:

$$\omega^2 r = \frac{ga^2}{r^2} \tag{5}$$

Let $r = a + h$ where $h$ is the height above the surface of the orbit, denote the period of the orbit as $\tau$, where $\tau = \frac{2\pi}{\omega}$. Show that

$$\tau^2 = \frac{(2\pi)^2(a+h)^3}{ga^2} = \frac{(2\pi)^2 a}{g}\left(1 + \frac{h}{a}\right)^3 \tag{6}$$

(b) Write a calculator or computer type formula for predicting the time to complete one orbit in units of hours, with the height input in units of kilometers. (Too difficult for a quiz with no calculator, but the method is quizzable.)

(c) Use your formula to find the number of hours in an orbit at 0 km, 200 km, and 1000 km above the surface. (Too difficult for a quiz with no calculator)

13. This is from the 2004 ps1.pdf. This is quizzable. Hint: use the distributive property of the vector and scalar product. There exist a variety of proofs for the identity

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) \tag{7}$$

Here we put together a proof based on an assumption (or fact) that the cross product, like the dot product, can be computed in any basis and yield the same result. This means that when we can write either

$$\vec{a} = a_x \widehat{\mathbf{\imath}} + a_y \widehat{\mathbf{\jmath}} + a_z \widehat{\mathbf{k}}, \tag{8}$$

or

$$\vec{a} = A_x \widehat{\mathbf{I}} + A_y \widehat{\mathbf{J}} + A_z \widehat{\mathbf{K}} \tag{9}$$

and similarly for $\vec{b}$ and $\vec{c}$, calculate a cross product *with the same formula* for either basis, and compare the result at the end (after expressing the resultant vector in a common basis), the result will be the same.

Suppose we choose to align $\widehat{\mathbf{\imath}}$ exactly with the $\vec{c}$, and choose to align $\widehat{\mathbf{\jmath}}$ such that $\vec{b}$, so that

$$\begin{aligned} \vec{c} &= c_x \widehat{\mathbf{\imath}} \qquad (10) \\ \vec{b} &= b_x \widehat{\mathbf{\imath}} + b_y \widehat{\mathbf{\jmath}} \\ \vec{a} &= a_x \widehat{\mathbf{\imath}} + a_y \widehat{\mathbf{\jmath}} + a_z \widehat{\mathbf{k}} \qquad (11) \end{aligned}$$

If we prove the identity for this basis, the identity should be true for any basis. Using this basis, calculate the left and right hand sides of the identity and show that equality holds. The proof is rather quick and simple, as compared to doing a "brute force" calculation for an arbitrary basis.

14. Again consider (4). A conservation of energy analysis is not particularly useful in this case, because we can be entirely successful in finding $w(t)$. Nevertheless, it is a useful exercise to show that (4) implies:

$$\frac{dE}{dt} = -kw^2 \tag{12}$$

where

$$E \equiv \frac{1}{2}v^2 + gz \tag{13}$$

For $k = 0$, $E$ is conserved. For $k > 0$, $E$ is lost. (Do you know where it goes?).

15. Consider the nonlinear spring:

$$m\frac{d^2x}{dt^2} = -kx^3 \tag{14}$$

Show the following $E$ is conserved:

$$E \equiv \frac{1}{2}mv^2 + \frac{1}{4}kx^4 \tag{15}$$

Suppose the mass is released from $x = L$ with $v = 0$ at $t = 0$. What is the value of $v$ at $x = 0$? Note: answering that question is much easier than answering *when* is the mass at $x = 0$.

16. Consider another nonlinear spring:

$$m\frac{dv}{dt} = -kL\sin\left(\frac{x}{L}\right) \tag{16}$$

where $\frac{dx}{dt} = v$. What quantity $E$ is conserved? Suppose the mass is released from $x = L$ with $v = 0$ at $t = 0$. What is the value of $v$ at $x = 0$?

17. We are done adding questions.