METR 3113: Atmospheric Dynamics 1

Equations, Units and Dimensions

Slide 1

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Slide 2

An equation from Stull's book, eqn. (8.18), the *medium spherical raindrop equation*:

$$w = \alpha \left(\frac{\rho_o}{\rho_{air}} R \right)^{1/2}$$

where w is the fallspeed, $\alpha = -220 \text{ m}^{1/2}\text{s}^{-1}$, $\rho_o = 1.225 \text{ kg m}^{-3}$ is air density at sealevel, ρ_{air} is the ambient density, R is the radius of the drop. Valid for $0.5 \text{ mm} < R < 1 \text{ mm}$.

Sealevel version ($\rho_{air} = \rho_o$):

$$w = \alpha R^{1/2}$$

Slide 3

Try $w = \alpha R^{1/2}$ with $R = 900\ \mu\text{m}$

$$\begin{aligned}
w &= -220 \text{ m}^{1/2}\text{s}^{-1}(900\mu\text{m})^{1/2} \\
&= -220 \text{ m}^{1/2}\text{s}^{-1}900^{1/2}\mu\text{m}^{1/2} \\
&= -220 \text{ m}^{1/2}\text{s}^{-1}30\mu\text{m}^{1/2} \\
&= -220 \times 30 \text{ m}^{1/2}\text{s}^{-1}\mu\text{m}^{1/2} \\
&= -220 \times 30 \text{ m}^{1/2}\mu\text{m}^{1/2}\text{s}^{-1} \\
&= -6600 \text{ m}^{1/2}\mu\text{m}^{1/2}\text{s}^{-1} \\
&= -6600 \text{ m}^{1/2}(10^{-6} \text{ m})^{1/2}\text{s}^{-1} \\
&= -6600 \text{ m}^{1/2}10^{-3} \text{ m}^{1/2}\text{s}^{-1} \\
&= -6600 \times 10^{-3}\text{m}^{1/2}\text{m}^{1/2}\text{s}^{-1} \\
&= -6.6 \text{ m s}^{-1}
\end{aligned}$$

Slide 4

Or convert the μm to m at the start:

$$R = 900\ \mu\text{m}\frac{1\text{m}}{10^6\mu\text{m}} = 9 \times 10^{-4} \text{ m}$$

$$\begin{aligned}
w &= -220 \text{ m}^{1/2}\text{s}^{-1}(9 \times 10^{-4}\text{m})^{1/2} \\
&= -220\ 3 \times 10^{-2} \text{ m}^{1/2}\text{s}^{-1}\text{m}^{1/2} \\
&= -6.6 \text{ m s}^{-1}
\end{aligned}$$

Or use both α and R in units of μm:

$$\begin{aligned} \alpha &= -220 \text{ m}^{1/2}\text{s}^{-1} \\ \alpha &= -220 \left(10^6 \mu\text{m}\right)^{1/2}\text{s}^{-1} \\ \alpha &= -2.2 \times 10^5 \ \mu\text{m}^{1/2}\text{s}^{-1} \end{aligned}$$

Convert to m at the end:

Slide 5

$$\begin{aligned} w &= -2.2 \times 10^5 \ \mu\text{m}^{1/2}\text{s}^{-1}(900\mu\text{m})^{1/2} \\ &= -2.2 \times 10^5 \ \mu\text{m}^{1/2}\text{s}^{-1}30\mu\text{m}^{1/2} \\ &= -6.6 \times 10^6 \ \mu\text{m s}^{-1} \\ &= -6.6 \times 10^6 \ \mu\text{m s}^{-1}\frac{10^{-6} \text{ m}}{\mu\text{m}} \\ &= -6.6 \text{ m s}^{-1} \end{aligned}$$

Note that

$$w = \alpha R^{1/2} \quad \text{with } \alpha = -220 \text{ m}^{1/2}\text{s}^{-1}$$

Slide 6

yields the same result independent of the units used for the computation.

Note the equation is dimensionally homogeneous.

Suppose we have many values of R for which we need to calculate w. Do we really want to keep track of the units again and again? Let's try this:

Slide 7

$$w = -220R^{1/2}$$

where R must be entered in meters and w will be output in units of meters per second.

But suppose the instructions are lost...

$$w = -220R^{1/2}$$

Try $R = 0.9$ mm (oops!):

$$w = -220\sqrt{0.9} = -220 \times 0.95 = -209$$

Slide 8

Suppose you *guess* that since mm is used as the units of R, then w will be output in mm s^{-1}.

$$w = -209 \text{ mm s}^{-1} \ ???$$

But

$$w = -209 \text{ mm s} \times \frac{1\text{m}}{1000\text{mm}} = -0.209 \text{ m s}^{-1} \neq -6.6 \text{ ms}^{-1}$$

$$w = -220R^{1/2}$$

Slide 9

Fast. No unit computations. But must use the correct implied units. Computers use this form. *So do real scientists.*

Slide 10

Suppose we wish to write computer code where droplet radius is input as DROPRAD in units of mm and FALLSPEED is output in units of m s^{-1}. What is the equation that should be programmed?

Slide 11

Begin with the universal form:

$$w = \alpha R^{1/2} \quad \text{with } \alpha = -220 \text{ m}^{1/2} \text{ s}^{-1}$$

Let:

$$w \equiv \text{FALLSPEED m s}^{-1}$$

$$R \equiv \text{DROPRAD mm}$$

$$\text{FALLSPEED m s}^{-1} = -220 \text{ m}^{1/2} \text{ s}^{-1}\text{DROPRAD}^{1/2} \text{ mm}^{1/2}$$

Slide 12

$$\text{FALLSPEED m s}^{-1} = -220 \text{ m}^{1/2} \text{ s}^{-1}\text{DROPRAD}^{1/2} \text{ mm}^{1/2}$$

$$\text{FALLSPEED} = -220 \text{ DROPRAD}^{1/2} \text{ m}^{-1/2} \text{ mm}^{1/2}$$

$$\text{FALLSPEED} = -220 \text{ DROPRAD}^{1/2} (1000\text{mm})^{-1/2} \text{ mm}^{1/2}$$

$$\text{FALLSPEED} = -220 \text{ DROPRAD}^{1/2} \, 1000^{-1/2} \text{ mm}^{-1/2} \text{ mm}^{1/2}$$

$$\text{FALLSPEED} = -6.96 \text{ DROPRAD}^{1/2}$$

$$\text{FALLSPEED} = -6.96 * \text{SQRT}(\text{DROPRAD})$$

Slide 13

Derive another unit-free formula

$$w = \alpha R^{1/2} \quad \text{with } \alpha = -220 \text{ m}^{1/2} \text{ s}^{-1}$$

Let:

$$w \equiv \text{FALLSPEED m s}^{-1}$$

$$R \equiv \text{DROPRAD m}$$

$$\text{FALLSPEED m s}^{-1} = -220 \text{ m}^{1/2} \text{ s}^{-1}\text{DROPRAD}^{1/2} \text{ m}^{1/2}$$

$$\text{FALLSPEED} = -220 \text{ DROPRAD}^{1/2}$$

Or more briefly:

$$w = -220 \; R^{1/2}$$

But note $w \neq w$.

Slide 14

in *computations*

$$w = -220R^{1/2}$$

prevails over

$$w = \alpha R^{1/2} \quad \text{with } \alpha = -220 \text{ m}^{1/2} \text{ s}^{-1}$$

But in *theoretical development* the latter is preferred.

Slide 15

For example, suppose you are asked to: *Find the fallspeed w of a medium raindrop in terms of its mass M.* The preferred derivation is:

$$M = \frac{4}{3}\pi\rho_w R^3$$

$$R = \left(\frac{3M}{4\pi\rho_w}\right)^{1/3}$$

$$w = \alpha R^{1/2}$$

$$w = \alpha \left(\frac{3}{4\pi\rho_w}\right)^{1/6} M^{1/6}$$

Slide 16

Working with $w = -220R^{1/2}$ is dangerous; we might assume $R = R$ and $w = w$, and end up with

$$w = -220 \left(\frac{3}{4\pi\rho_w}\right)^{1/6} M^{1/6}$$

instead of

$$w = \alpha \left(\frac{3}{4\pi\rho_w}\right)^{1/6} M^{1/6}$$