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METR 3113: Atmospheric Dynamics 1

Why Vectors are Useful

Lecture for Monday, August 27, 2007
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Our math culture allows binary operations on elements other than numbers.

$$\begin{aligned} 3+2 &= 5 \\ 3\ \mathrm{kg}+2\ \mathrm{kg} &= 5\ \mathrm{kg} \\ 3\ \mathrm{ft}+2\ \mathrm{in} &= 38\ \mathrm{in} \\ 3\ \mathrm{kg}+2\ \mathrm{in} &= \mathrm{ERROR} \\ 3\hat{\mathbf{\imath}}+2\hat{\mathbf{\jmath}} &= 3\hat{\mathbf{\imath}}+2\hat{\mathbf{\jmath}} \end{aligned}$$

Why add things together that are not reducible?

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Consider this:

$$1 \text{ sandwhich} = 2 \text{ bread} + 1 \text{ salami}$$

It might be useful to bundle together a list with "+" signs because it provides easy consistency with the rule that multiplication is distributive over the sum:

$$\begin{aligned} 3 \times 1 \text{ sandwhich} &= 3 \times (2 \text{ bread} + 1 \text{ salami}) \\ 3 \times 1 \text{ sandwhich} &= 6 \text{ bread} + 2 \text{ salami} \end{aligned}$$

...possibly useful for inventory control in a deli.

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Another example: A cockroach has velocity:

$$\vec{v} = 4\ \mathrm{cm\ s^{-1}}\ \hat{\mathbf{\imath}} + 2\ \mathrm{cm\ s^{-1}}\ \hat{\mathbf{\jmath}}$$

How far does the cockroach travel to $\Delta t = 2\mathrm{s}$? Answer:

$$\begin{aligned} \vec{d} &= \Delta t\ \vec{v} \\ &= 2\ \mathrm{s} \left(4\ \mathrm{cm\ s^{-1}}\ \hat{\mathbf{\imath}} + 2\ \mathrm{cm\ s^{-1}}\ \hat{\mathbf{\jmath}}\right) \\ &= 8\ \mathrm{cm}\ \hat{\mathbf{\imath}} + 4\ \mathrm{cm}\ \hat{\mathbf{\jmath}} \end{aligned}$$

The unit vectors and arrow symbols don't add much efficiency. A verbal method is satisfactory: A cockroach walks at 4 cm s^{-1} toward the east and 2 cm s^{-1} toward the north. How far...

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Vector mathematics becomes very efficient with the use of products the *scalar (dot) product* $\vec{a} \cdot \vec{b}$ and the *vector (cross) product* $\vec{a} \times \vec{b}$.

For example, recall the force $\vec{F}$ on a particle with charge q and velocity $\vec{v}$ as caused by a magnetic field $\vec{B}$:

$$\vec{F} = q\vec{v} \times \vec{B}$$

In meteorology we have the law for the Coriolis force $\vec{F_c}$:

$$\frac{1}{m}\vec{F_c} = \vec{v} \times 2\vec{\Omega}$$

where m is the mass of the blob, $\vec{v}$ is the velocity vector of the blob and $\vec{\Omega}$ is the rotation vector for Earth.

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Another example from meteorology: the rate of change of kinetic energy of a hailstone, in response to gravity $\vec{g}$ and the drag force $\vec{F_d}$:

$$\frac{d}{dt}\frac{mv^2}{2} = \vec{v} \cdot \vec{g}\; m + \vec{v} \cdot \vec{F_d}$$

In future courses you will learn how to derive the incompressible vorticity equation:

$$\frac{\partial \vec{\omega}}{\partial t} + \vec{U} \cdot \nabla \vec{\omega} = \vec{\omega} \cdot \nabla \vec{U} + \frac{1}{\rho^2} \nabla \rho \times \nabla p$$

where $\vec{\omega} \equiv \nabla \times \vec{U}$.

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Vector equations are valid independent of the choice of direction for $\widehat{\mathbf{\imath}}$, $\widehat{\mathbf{\jmath}}$ and $\widehat{\mathbf{k}}$.

Analogous to the fact that an equation like $w = \alpha R^{1/2}$ is true independent of the choice of units.

The previous vector equations are both dimensionally homogeneous and orientationally invariant.

Note: orientational invariance is also commonly referred to as rotational invariance.

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We need an agreement for what we mean by expressing a vector $\vec{U}$ in an *alternatively-oriented coordinate system.*

This is analogous to our agreement about how to express a speed such as w in *alternative units.*

First we consider the how to express the position of a point in a plane, the rule for position vectors and all vector then follows.

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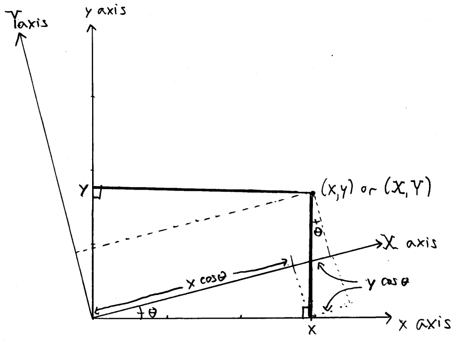

$$X = x\ \cos(\theta) + y\ \sin(\theta)$$

$$Y = -x\ \sin(\theta) + y\ \cos(\theta)$$

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The reciprocal relations can be derived without recourse to a picture:

$$X\ \cos(\theta) = x\ \cos^2(\theta) + y\ \sin(\theta)\cos(\theta)$$

$$Y\ \sin(\theta) = -x\ \sin^2(\theta) + y\ \cos(\theta)\sin(\theta)$$

$$X\ \cos(\theta) - Y\ \sin(\theta) = x\left[\cos^2(\theta) + \sin^2(\theta)\right]$$

$$x = X\ \cos(\theta) - Y\ \sin(\theta)$$

Similarly

$$y = X\ \sin(\theta) + Y\ \cos(\theta)$$

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Rotate the 2-d position vector:

$$\vec{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}}$$

$$\vec{r} = [X\ \cos(\theta) - Y\ \sin(\theta)]\,\hat{\mathbf{\imath}} + [X\ \sin(\theta) + Y\ \cos(\theta)]\,\hat{\mathbf{\jmath}}$$

$$\vec{r} = X\,[\cos(\theta)\hat{\mathbf{\imath}} + \sin(\theta)\hat{\mathbf{\jmath}}] + Y\,[-\sin(\theta)\hat{\mathbf{\imath}} + \cos(\theta)\hat{\mathbf{\jmath}}]$$

Or let

$$\hat{\mathbf{I}} \equiv \cos(\theta)\hat{\mathbf{\imath}} + \sin(\theta)\hat{\mathbf{\jmath}}$$

$$\hat{\mathbf{J}} \equiv -\sin(\theta)\hat{\mathbf{\imath}} + \cos(\theta)\hat{\mathbf{\jmath}}$$

So

$$\vec{r} = X\hat{\mathbf{I}} + Y\hat{\mathbf{J}}$$

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Generalize to vectors:

$$\vec{a} = a_x\hat{\mathbf{\imath}} + a_y\hat{\mathbf{\jmath}} = a_X\hat{\mathbf{I}} + a_Y\hat{\mathbf{J}}$$

will be true with

$$a_X = a_x\cos(\theta) + a_y\sin(\theta)$$

$$a_Y = -a_x\sin(\theta) + a_y\cos(\theta)$$

and

$$\hat{\mathbf{I}} \equiv \cos(\theta)\hat{\mathbf{\imath}} + \sin(\theta)\hat{\mathbf{\jmath}}$$

$$\hat{\mathbf{J}} \equiv -\sin(\theta)\hat{\mathbf{\imath}} + \cos(\theta)\hat{\mathbf{\jmath}}$$

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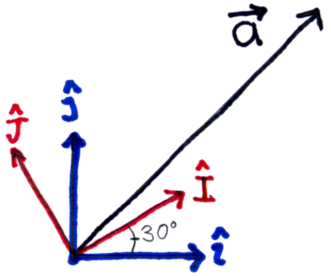

$$\cos(\theta) = \cos(30°) = \frac{\sqrt{3}}{2}$$

$$\sin(\theta) = \sin(30°) = \frac{1}{2}$$

$$\vec{a} = 2\widehat{\mathbf{\imath}} + 2\widehat{\mathbf{\jmath}}$$

$$a_x = 2 \text{ and } a_y = 2$$

$$a_X = a_x \cos(\theta) + a_y \sin(\theta) = 2\frac{\sqrt{3}}{2} + 2\frac{1}{2} = \sqrt{3} + 1$$

$$a_Y = -a_x \sin(\theta) + a_y \cos(\theta) = -2\frac{1}{2} + 2\frac{\sqrt{3}}{2} = -1 + \sqrt{3}$$

$$\vec{a} = (\sqrt{3} + 1)\widehat{\mathbf{I}} + (-1 + \sqrt{3})\widehat{\mathbf{J}}$$

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Another example vector (quite simple):

$$\vec{b} = 1\widehat{\mathbf{\imath}} + 0\widehat{\mathbf{\jmath}}$$

$$b_X = b_x \cos(\theta) + b_y \sin(\theta) = \frac{\sqrt{3}}{2}$$

$$b_Y = -b_x \sin(\theta) + b_y \cos(\theta) = -\frac{1}{2}$$

$$\vec{b} = \frac{\sqrt{3}}{2}\widehat{\mathbf{I}} - \frac{1}{2}\widehat{\mathbf{J}}$$

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Using what you know already about the dot product, try $\vec{a} \cdot \vec{b}$ with the different bases:

$$\vec{a} \cdot \vec{b} = (2\widehat{\mathbf{\imath}} + 2\widehat{\mathbf{\jmath}}) \cdot (1\widehat{\mathbf{\imath}} + 0\widehat{\mathbf{\jmath}}) = 2 \cdot 1 + 2 \cdot 0 = 2$$

$$\begin{aligned} \vec{a} \cdot \vec{b} &= \left[(\sqrt{3} + 1)\widehat{\mathbf{I}} + (-1 + \sqrt{3})\widehat{\mathbf{J}}\right] \cdot \left(\frac{\sqrt{3}}{2}\widehat{\mathbf{I}} - \frac{1}{2}\widehat{\mathbf{J}}\right) \\ &= (\sqrt{3} + 1)\frac{\sqrt{3}}{2} + (-1 + \sqrt{3})\left(-\frac{1}{2}\right) \\ &= \frac{\sqrt{3}^2}{2} + \frac{1}{2} = \frac{3}{2} + \frac{1}{2} = 2 \end{aligned}$$

So $\vec{a} \cdot \vec{b} = 2$ in either coordinate system. Will that always be true for the dot product?