METR 3113: Atmospheric Dynamics 1

More Equation of Motion in 1-D

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Lecture for Monday, September 17, 2007

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$$\boxed{a = a(x)}$$

Common in modeling the natural world and in engineering.

Archetypal equation is mass on a spring:

$$m\frac{dv}{dt} = -kx$$

m and k are constants and

$$v \equiv \frac{dx}{dt}$$

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Write as a second-order O.D.E. for $x(t)$:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

with

$$\omega^2 \equiv \frac{k}{m}$$

We should recall from memory the general solution:

$$x(t) = A\cos(\omega t) + B\sin(\omega t)$$

The constants A and B could be determined from initial conditions $x(0)$ and $\frac{dx}{dt}(0)$.

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$$x(0) = A\cos(\omega 0) + B(\sin\omega 0) = A$$

$$\frac{dx}{dt} = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

$$\frac{dx}{dt}(0) = -\omega A\sin(\omega 0) + \omega B\cos(\omega 0)$$

$$\frac{dx}{dt}(0) = \omega B$$

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$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

May arise when there is a slight imbalance between two large, opposing forces.

Let z be the vertical coordinate and w be the upward velocity component. Let $z = 0$ be the hook on the end of the spring, before a mass is attached. The equation of motion is:

$$m\frac{d^2z}{dt^2} = -mg - kz$$

Let $x(t)$ be the vertical displacement from the equilibrium position z_e:

$$0 = -mg - kz_e$$

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Or

$$z_e = -\frac{mg}{k}$$

We write

$$z(t) = z_e + x(t)$$

Substitution into:

$$m\frac{d^2z}{dt^2} = -mg - kz$$

Gives

$$m\frac{d^2}{dt^2}\left(z_e + x\right) = -mg - k\left(z_e + x\right)$$

which leaves an equation independent from g:

$$m\frac{d^2x}{dt^2} = -kx$$

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Here is another example of a=a(x).

A comet moving on a straight-line path to or from the sun could obey:

$$\frac{dv}{dt} = -\frac{GM}{x^2}$$

which is the same as

$$\frac{d^2x}{dt^2} = -\frac{GM}{x^2}\ .$$

The solution of this equation for $x(t)$ is beyond the scope of this course.

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Easier to extract information about $v(x)$ rather than $x(t)$.

Multiply all terms by v, or its equivalent $\frac{dx}{dt}$:

$$v\frac{dv}{dt} = -\frac{GM}{x^2}\frac{dx}{dt}$$

Using the chain rule, this can be written as

$$\frac{d}{dt}\frac{v^2}{2} = \frac{d}{dt}\frac{GM}{x}$$

Or

$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{GM}{x}\right) = 0$$

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$$\frac{d}{dt}\left(\frac{v^2}{2} - \frac{GM}{x}\right) = 0$$

implies

$$\frac{v^2}{2} - \frac{GM}{x} = E = \text{constant}$$

The *energy per unit mass* E remains invariant. The relation $v(x)$ implied in the above allows for a number of useful calculations to be made.

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Consider two values of x on the trajectory, x_1 and x_2. Let $v_1 \equiv v(x_1)$ and $v_2 \equiv v(x_2)$. With both

$$\frac{v_1^2}{2} - \frac{GM}{x_1} = E$$

and

$$\frac{v_2^2}{2} - \frac{GM}{x_2} = E$$

we have

$$\frac{v_1^2}{2} - \frac{GM}{x_1} = \frac{v_2^2}{2} - \frac{GM}{x_2}$$

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For a comet starting from $x_1 \to \infty$ with $v_1 \to 0$ and impacting a planet of radius $x_2 = R$:

$$0 + 0 = \frac{v_2^2}{2} - \frac{GM}{R}.$$

Or

$$\frac{v_2^2}{2} = \frac{GM}{R}$$

There is no requirement that position x_1 precedes position x_2 in time.

v_2 could also be the velocity required for a projectile shot from an isolated planet to coast off to infinity, with the velocity gradually becoming vanishingly small.

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The spring again. Multiply both sides of the spring equation by v or its equivalent:

$$mv\frac{dv}{dt} = -kx\frac{dx}{dt}$$

$$\frac{d}{dt}\left(\frac{1}{2}mv^2 + \frac{1}{2}kx^2\right) = 0$$

This implies

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E = \text{constant}$$

where E is now *energy*. E is known from initial conditions $x(0)$ and $\frac{dx}{dt}(0)$.

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Even without knowing trig functions can be used to solve for $x(t)$, we could use

$$\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = E = \text{constant}$$

to solve for things like:

- the maximum value of $|v|$, which occurs when $x = 0$
- the maximum value of $|x|$, which occurs when $v = 0$

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$$\boxed{a = a(v, x)}$$

The damped harmonic oscillator:

$$m\frac{dv}{dt} = -bv - kx$$

where b and k are constants.

First derive an energy equation.

May help us to find mistakes in proposed solutions for $v(t)$ and $x(t)$.

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$$mv\frac{dv}{dt} = -bvv - kx\frac{dx}{dt}$$

or

$$\frac{dE}{dt} = -bv^2$$

where E is again the sum of kinetic and potential energy:

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 \ .$$

With $b > 0$, E will decrease whenever the object is moving, since $v^2 \geq 0$.

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$$m\frac{dv}{dt} + bv + kx = 0$$

can be written

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

where

$$\gamma \equiv \frac{b}{2m}$$

$$\omega_o^2 \equiv \frac{k}{m}$$

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Finding $x(t)$. We *propose* a solution

$$x(t) = Ae^{-\sigma t}\cos(\omega t + \phi)$$

$$(\omega_0^2 - 2\gamma\sigma + \sigma^2 - \omega^2)e^{-\sigma t}\cos(\omega t + \phi) + (2\sigma\omega - 2\gamma\omega)e^{-\sigma t}\sin(\omega t + \phi) = 0$$

This will be true for all values of t only if

$$\begin{aligned} 2\sigma\omega - 2\gamma\omega &= 0 \\ \omega_0^2 - 2\gamma\sigma + \sigma^2 - \omega^2 &= 0 \end{aligned}$$

The above have a solution

$$\begin{aligned} \sigma &= \gamma \\ \omega^2 &= \omega_o^2 - \gamma^2 \end{aligned}$$

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Provided that $\omega_o^2 - \gamma^2 > 0$, (meaning the damping is small compared to the spring force), a solution to

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_o^2 x = 0$$

is

$$x(t) = Ae^{-\gamma t}\cos(\omega t + \phi)$$

with

$$\omega = \sqrt{\omega_o^2 - \gamma^2}$$

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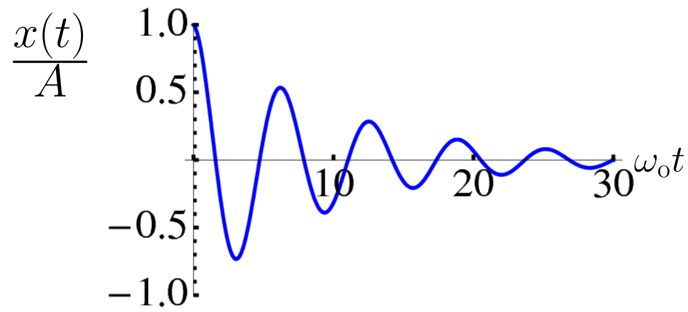

$$x(t) = Ae^{-\gamma t}\cos(\omega t + \phi)$$

with $\gamma = .1\omega_o$, $\omega = .9950\omega_o$ and $\phi = 0$