Slide 1

METR 3113: Atmospheric Dynamics 1

Choosing A Coordinate System

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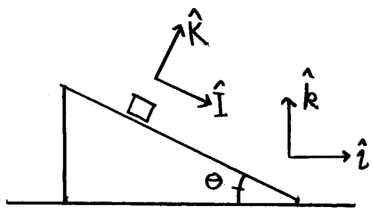
Slide 2



$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + \vec{N}$$

$$m\frac{d^2\vec{r}}{dt^2} = -mg\widehat{\mathbf{k}} + N\widehat{\mathbf{K}}$$

To solve for the time required to slide down the frictionless plane a distance L, what co-ordinate system should be used?

Slide 3

$$\widehat{\mathbf{I}} = \cos\theta\,\widehat{\mathbf{\imath}} - \sin\theta\,\widehat{\mathbf{k}}$$

$$\widehat{\mathbf{K}} = \sin\theta\,\widehat{\mathbf{\imath}} + \cos\theta\,\widehat{\mathbf{k}}$$

$$\vec{r} = x\widehat{\mathbf{\imath}} + z\widehat{\mathbf{k}}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\widehat{\mathbf{\imath}} + \frac{d^2z}{dt^2}\widehat{\mathbf{k}}$$

$$\vec{r} = X\widehat{\mathbf{I}} + Z\widehat{\mathbf{K}}$$

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2X}{dt^2}\widehat{\mathbf{I}} + \frac{d^2Z}{dt^2}\widehat{\mathbf{K}}$$

Slide 4

The equation of motion in the $\widehat{\mathbf{I}}$ direction is:

$$\widehat{\mathbf{I}}\cdot\frac{d^2\vec{r}}{dt^2} = -g\widehat{\mathbf{k}}\cdot\widehat{\mathbf{I}} + \frac{N}{m}\widehat{\mathbf{K}}\cdot\widehat{\mathbf{I}}$$

$$\frac{d^2X}{dt^2} = g\sin\theta$$

We can easily solve for the time to accelerate from rest at $X = -L$ and travel to $X = 0$.

Slide 5

The equation of motion in the $\hat{\imath}$ direction:

$$\hat{\imath} \cdot \frac{d^2\vec{r}}{dt^2} = -g\hat{\mathbf{k}} \cdot \hat{\imath} + \frac{N}{m}\hat{\mathbf{K}} \cdot \hat{\imath}$$

$$\frac{d^2x}{dt^2} = \frac{N}{m}\sin\theta$$

This cannot be solved without knowledge of N.

Slide 6

The equation of motion in the $\hat{\mathbf{k}}$ direction:

$$\hat{\mathbf{k}} \cdot \frac{d^2\vec{r}}{dt^2} = -g\hat{\mathbf{k}} \cdot \hat{\mathbf{k}} + \frac{N}{m}\hat{\mathbf{K}} \cdot \hat{\mathbf{k}}$$

or

$$\frac{d^2z}{dt^2} = -g + \frac{N}{m}\cos\theta$$

Again we see N, still unknown.

Slide 7

N can be eliminated between

$$\frac{d^2x}{dt^2} = \frac{N}{m}\sin\theta$$

$$\frac{d^2z}{dt^2} = -g + \frac{N}{m}\cos\theta$$

$$\frac{d^2z}{dt^2} = -g + \frac{\cos\theta}{\sin\theta}\frac{d^2x}{dt^2}$$

... N is gone, but we still have both x and z.

Slide 8

Use kinematical constraint: the block slides down the plane and does not bounce.

The block must follow the slope of the plane:

$$dz = -dx\tan\theta$$

$$\frac{d^2z}{dt^2} = -g + \frac{\cos\theta}{\sin\theta}\frac{d^2x}{dt^2}$$

becomes

$$-\tan\theta\frac{d^2x}{dt^2} = -g + \frac{\cos\theta}{\sin\theta}\frac{d^2x}{dt^2}$$

$$-\frac{\sin\theta}{\cos\theta}\frac{d^2x}{dt^2} = -g + \frac{\cos\theta}{\sin\theta}\frac{d^2x}{dt^2}$$

$$-\sin^2\theta\,\frac{d^2x}{dt^2} = -g\sin\theta\ \cos\theta\ + \cos^2\theta\,\frac{d^2x}{dt^2}$$

Slide 9

$$\frac{d^2x}{dt^2} = -g\sin\theta\ \cos\theta$$

Starting at position of $X = -L$ corresponds to $x = -L\ \cos\theta$.

You can show the time to move a distance $\Delta x = L\ \cos\theta$ is the same time to move $\Delta X = L$.

Slide 10

$\vec{N}$ appears to push the object to the right in:

$$\frac{d^2x}{dt^2} = \frac{N}{m}\sin\theta$$

But $\vec{N}$ has no role in:

$$\frac{d^2X}{dt^2} = g\sin\theta$$

So is $\vec{N}$ important or not? What does it do? An energy equation helps to answer that question, independent of the coordinate system.

Slide 11

Our first example of an energy equation derived from a vector equation of motion:

$$m\frac{d^2\vec{r}}{dt^2} = m\vec{g} + \vec{N}$$

$$\vec{v}\cdot m\frac{d\vec{v}}{dt} = \vec{v}\cdot m\vec{g} + \vec{v}\cdot\vec{N}$$

But $\vec{v}\cdot\vec{N} = 0$, because the $\vec{v}$ is tangent to the plane and $\vec{N}$ is normal to it.

$$\vec{v}\cdot\vec{g} = \left(\frac{dx}{dt}\hat{\imath} + \frac{dz}{dt}\hat{\mathbf{k}}\right)\cdot\left(-g\hat{\mathbf{k}}\right) = -g\frac{dz}{dt}$$

Slide 12

Also, you can prove:

$$\frac{d}{dt}\left(\frac{1}{2}\vec{v}\cdot\vec{v}\right) = \vec{v}\cdot\frac{d}{dt}\vec{v}\ .$$

So

$$\frac{d}{dt}\left(\frac{1}{2}m\vec{v}\cdot\vec{v}\right) = \vec{v}\cdot m\vec{g} = -m\frac{dz}{dt}g$$

Or

$$\frac{dE}{dt} = 0$$

with

$$E \equiv \left(\frac{1}{2}mv^2 + mgz\right)$$

$\vec{N}$ plays no role in the energy equation.

circular motion

Slide 13

Cartesian position vector in a plane:

$$\vec{r} = x\hat{\mathbf{\imath}} + y\hat{\mathbf{\jmath}}$$

Conventional relation between polar coordinates (r, θ) and Cartesian coordinates (x, y):

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}$$

A "mixed" representation of position:

$$\vec{r} = r\cos\theta\,\hat{\mathbf{\imath}} + r\sin\theta\,\hat{\mathbf{\jmath}}$$

Slide 14

Suppose a particle moves in a circle with $r = R$, with constant angular velocity ω, or $\theta = \omega t$:

$$\vec{r} = R\cos(\omega t)\,\hat{\mathbf{\imath}} + R\sin(\omega t)\,\hat{\mathbf{\jmath}}$$

Note $|\vec{r}| = R$.

With R, ω, $\hat{\mathbf{\imath}}$ and $\hat{\mathbf{\jmath}}$ assumed to be independent of time,

$$\vec{v} = \frac{d\vec{r}}{dt} = -R\omega\sin(\omega t)\,\hat{\mathbf{\imath}} + R\omega\cos(\omega t)\,\hat{\mathbf{\jmath}}$$

Note $|\vec{v}| = v = \omega R$ and $\vec{v}\cdot\vec{r} = 0$.

$$\vec{a} = \frac{d\vec{v}}{dt} = -R\omega^2\cos(\omega t)\,\hat{\mathbf{\imath}} - R\omega^2\sin(\omega t)\,\hat{\mathbf{\jmath}}$$

$$\vec{a} = -\omega^2\vec{r} = -\frac{v^2}{r^2}\vec{r}$$

Slide 15

Define $\hat{\mathbf{r}}$:

$$\vec{r} = r\hat{\mathbf{r}}$$

$$\vec{a} = -\omega^2 r\hat{\mathbf{r}} = -\frac{v^2}{r}\hat{\mathbf{r}}$$

In uniform circular motion, the acceleration is directed opposite the direction pointing outward to the object, meaning toward the center, with magnitude $\omega^2 r$, or $\frac{v^2}{r}$.

This is the centripetal acceleration.

Slide 16

For the record:

$$\hat{\mathbf{r}} = \frac{\vec{r}}{r} = \cos\theta\,\hat{\mathbf{\imath}} + \sin\theta\,\hat{\mathbf{\jmath}}$$

Recall for uniform circular motion with $\theta = \omega t$:

$$\vec{v} = -v\sin(\omega t)\,\hat{\mathbf{\imath}} + v\cos(\omega t)\,\hat{\mathbf{\jmath}} = v\hat{\boldsymbol{\theta}}$$

where $\hat{\boldsymbol{\theta}}$ naturally appears

$$\hat{\boldsymbol{\theta}} \equiv -\sin\theta\,\hat{\mathbf{\imath}} + \cos\theta\,\hat{\mathbf{\jmath}}$$

Slide 17

Next we will explore "doing dynamics" entirely in the polar coordinate basis $\widehat{\mathbf{r}}$ and $\widehat{\boldsymbol{\theta}}$. We can use the Cartesian definitions

$$\widehat{\mathbf{r}} \equiv \cos\theta\,\widehat{\mathbf{\imath}} + \sin\theta\,\widehat{\mathbf{\jmath}} \qquad \widehat{\boldsymbol{\theta}} \equiv -\sin\theta\,\widehat{\mathbf{\imath}} + \cos\theta\,\widehat{\mathbf{\jmath}}$$

to establish:

$$\frac{d\widehat{\mathbf{r}}}{d\theta} = \widehat{\boldsymbol{\theta}} \qquad \frac{d\widehat{\boldsymbol{\theta}}}{d\theta} = -\widehat{\mathbf{r}}$$

and

$$\frac{d\widehat{\mathbf{r}}}{dt} = \frac{d\theta}{dt}\widehat{\boldsymbol{\theta}} \qquad \frac{d\widehat{\boldsymbol{\theta}}}{dt} = -\frac{d\theta}{dt}\widehat{\mathbf{r}}$$

But after the derivation, we leave the Cartesian coordinates and trig functions behind.