1:

METR 3113: Atmospheric Dynamics 1

More About the 2-d Equation of Motion

Lecture for Friday, November 2, 2007
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2:

Recall the 2-D equation of motion:

$$\frac{d\vec{V}}{dt} = f\vec{V} \times \widehat{\mathbf{k}} - \frac{1}{\rho}\nabla_{H} p$$

or

$$\frac{d\vec{V}}{dt} = f\vec{V} \times \widehat{\mathbf{k}} - g_o \nabla_{p} Z$$

BTW, you may have seen me write $\nabla_p Z$ as $\nabla_H Z$, but $\nabla_p Z$ is a better idea ... can you explain why?

3:

Derive an energy equation:

$$\vec{V} \cdot \frac{d\vec{V}}{dt} = \vec{V} \cdot f\vec{V} \times \widehat{\mathbf{k}} - \vec{V} \cdot g_o \nabla_{p} Z$$

$$\frac{d}{dt}\left(\frac{\vec{V}\cdot\vec{V}}{2}\right) = \frac{d}{dt}\left(\frac{V^2}{2}\right) = -\vec{0} - \vec{V} \cdot g_o \nabla_{p} Z$$

The Coriolis force can only change the direction of the wind, it cannot change the speed of the wind.

Flow from high to low (heights or pressure) will increase the speed V.

4:

Let's review how to apply

$$\frac{d\vec{V}}{dt} = f\vec{V} \times \widehat{\mathbf{k}} - g_o \nabla_{p} Z$$

within a specific coordinate sytem. Consider a Cartesian coordinate system with $\widehat{\mathbf{\imath}}$ directed toward the east and $\widehat{\mathbf{\jmath}}$ directed toward the north and write $\vec{V} = u\widehat{\mathbf{\imath}} + v\widehat{\mathbf{\jmath}}$.

$$\frac{d}{dt}\left(u\widehat{\mathbf{\imath}} + v\widehat{\mathbf{\jmath}}\right) = f\left(u\widehat{\mathbf{\imath}} + v\widehat{\mathbf{\jmath}}\right) \times \widehat{\mathbf{k}} - g_o\frac{\partial Z}{\partial x}\widehat{\mathbf{\imath}} - g_o\frac{\partial Z}{\partial y}\widehat{\mathbf{\jmath}}$$

It is tempting to write

$$\frac{d}{dt}\left(u\widehat{\mathbf{\imath}} + v\widehat{\mathbf{\jmath}}\right) = \frac{du}{dt}\widehat{\mathbf{\imath}} + \frac{dv}{dt}\widehat{\mathbf{\jmath}}$$

but recall there are issues...

5:

- Independent of the fact that the Earth is rotating (a fact already accounted for in the Coriolis term $f\vec{V} \times \widehat{\mathbf{k}}$) the directions $\widehat{\mathbf{\imath}}$ and $\widehat{\mathbf{\jmath}}$ vary with position on the Earth. For example, with ϕ latitude, $\frac{\partial}{\partial \phi}\widehat{\mathbf{\jmath}} = -\widehat{\mathbf{k}}$
- Neglecting such derivatives is the same as neglecting the radius of the curvature of flow as caused by the sphericity of the Earth, which would be of order the radius of the Earth, or 6370 km.
- The radius of curvature of large-scale flow is of order 1000 km. The value is not significantly altered by accounting for the Earth's sphericity.

6:

Thus, for many applications, we are justified in making a *planar approximation*:

$$\frac{d}{dt}\left(u\widehat{\mathbf{\imath}} + v\widehat{\mathbf{\jmath}}\right) = \frac{du}{dt}\widehat{\mathbf{\imath}} + \frac{dv}{dt}\widehat{\mathbf{\jmath}}$$

$$\frac{d\vec{V}}{dt} = f\vec{V} \times \widehat{\mathbf{k}} - g_{\mathrm{o}}\nabla_{p} Z$$

$$\frac{d}{dt}\left(u\widehat{\mathbf{\imath}} + v\widehat{\mathbf{\jmath}}\right) = f\left(u\widehat{\mathbf{\imath}} + v\widehat{\mathbf{\jmath}}\right) \times \widehat{\mathbf{k}} - g_{\mathrm{o}}\frac{\partial Z}{\partial x}\widehat{\mathbf{\imath}} - g_{\mathrm{o}}\frac{\partial Z}{\partial y}\widehat{\mathbf{\jmath}}$$

becomes, with the planar approximation,

$$\frac{du}{dt}\widehat{\mathbf{\imath}} + \frac{dv}{dt}\widehat{\mathbf{\jmath}} = -fu\widehat{\mathbf{\jmath}} + fv\widehat{\mathbf{\imath}} - g_{\mathrm{o}}\frac{\partial Z}{\partial x}\widehat{\mathbf{\imath}} - g_{\mathrm{o}}\frac{\partial Z}{\partial y}\widehat{\mathbf{\jmath}}$$

$$\begin{aligned} \frac{du}{dt} &= +fv - g_{\mathrm{o}}\frac{\partial Z}{\partial x} \\ \frac{dv}{dt} &= -fu - g_{\mathrm{o}}\frac{\partial Z}{\partial y} \end{aligned}$$

7:

Continuing with the planar approximation, let's write $\vec{V}$ in plane polar coordinates:

$$\vec{V} = v_r\widehat{\mathbf{r}} + v_\theta\widehat{\boldsymbol{\theta}}$$

$$\frac{d\vec{V}}{dt} = f\vec{V} \times \widehat{\mathbf{k}} - g_{\mathrm{o}}\nabla_{p} Z$$

$$\frac{d}{dt}\left(v_r\widehat{\mathbf{r}} + v_\theta\widehat{\boldsymbol{\theta}}\right) = f\left(v_r\widehat{\mathbf{r}} + v_\theta\widehat{\boldsymbol{\theta}}\right) \times \widehat{\mathbf{k}} - g_{\mathrm{o}}\frac{\partial Z}{\partial r}\widehat{\mathbf{r}} - g_{\mathrm{o}}\frac{1}{r}\frac{\partial Z}{\partial \theta}\widehat{\boldsymbol{\theta}}$$

$$\begin{aligned} \frac{d}{dt}\left(v_r\widehat{\mathbf{r}} + v_\theta\widehat{\boldsymbol{\theta}}\right) &= \\ -fv_r\widehat{\boldsymbol{\theta}} \quad &+ \quad fv_\theta\widehat{\mathbf{r}} - g_{\mathrm{o}}\frac{\partial Z}{\partial r}\widehat{\mathbf{r}} - g_{\mathrm{o}}\frac{1}{r}\frac{\partial Z}{\partial \theta}\widehat{\boldsymbol{\theta}} \end{aligned}$$

But don't assume this:

$$\frac{d}{dt}\left(v_r\widehat{\mathbf{r}} + v_\theta\widehat{\boldsymbol{\theta}}\right) = \frac{dv_r}{dt}\widehat{\mathbf{r}} + \frac{dv_\theta}{dt}\widehat{\boldsymbol{\theta}}$$

8:

$$\frac{d}{dt}\left(v_r\widehat{\mathbf{r}} + v_\theta\widehat{\boldsymbol{\theta}}\right) = \frac{dv_r}{dt}\widehat{\mathbf{r}} + \frac{dv_\theta}{dt}\widehat{\boldsymbol{\theta}} + v_r\frac{d\widehat{\mathbf{r}}}{dt} + v_\theta\frac{d\widehat{\boldsymbol{\theta}}}{dt}$$

$$\frac{d}{dt}\left(v_r\widehat{\mathbf{r}} + v_\theta\widehat{\boldsymbol{\theta}}\right) = \left(\frac{dv_r}{dt} - \frac{v_\theta^2}{r}\right)\widehat{\mathbf{r}} + \left(\frac{dv_\theta}{dt} + \frac{v_\theta v_r}{r}\right)\widehat{\boldsymbol{\theta}}$$

$$\begin{aligned} \left(\frac{dv_r}{dt} - \frac{v_\theta^2}{r}\right)\widehat{\mathbf{r}} \quad &+ \quad \left(\frac{dv_\theta}{dt} + \frac{v_\theta v_r}{r}\right)\widehat{\boldsymbol{\theta}} \\ &= \quad -fv_r\widehat{\boldsymbol{\theta}} + fv_\theta\widehat{\mathbf{r}} - g_{\mathrm{o}}\frac{\partial Z}{\partial r}\widehat{\mathbf{r}} - g_{\mathrm{o}}\frac{1}{r}\frac{\partial Z}{\partial \theta}\widehat{\boldsymbol{\theta}} \end{aligned}$$

$$\frac{dv_r}{dt} - \frac{v_\theta^2}{r} = fv_\theta - g_{\mathrm{o}}\frac{\partial Z}{\partial r}$$

$$\frac{dv_\theta}{dt} + \frac{v_\theta v_r}{r} = -fv_r - g_{\mathrm{o}}\frac{1}{r}\frac{\partial Z}{\partial \theta}$$

As you know, if the origin to the polar coordinate system is chosen properly, $v_r = 0$ and the radial component is the equation for *gradient wind balance*:

$$-\frac{v_\theta^2}{r} = f v_\theta - g_\mathrm{o}\frac{\partial Z}{\partial r}$$

An important equation indeed!

The azimuthal component

$$\frac{dv_\theta}{dt} + \frac{v_\theta v_r}{r} = -f v_r - g_\mathrm{o}\frac{1}{r}\frac{\partial Z}{\partial \theta}$$

can be written

$$\frac{dv_\theta}{dt} + \frac{v_\theta}{r}\frac{dr}{dt} = -f\frac{dr}{dt} - g_\mathrm{o}\frac{1}{r}\frac{\partial Z}{\partial \theta}$$

$$r\frac{dv_\theta}{dt} + v_\theta\frac{dr}{dt} + fr\frac{dr}{dt} = -g_\mathrm{o}\frac{\partial Z}{\partial \theta}$$

$$\frac{d}{dt}\left(r v_\theta + \frac{1}{2}fr^2\right) = -g_\mathrm{o}\frac{\partial Z}{\partial \theta}$$

When $\frac{\partial Z}{\partial \theta} = 0$, we have conservation of angular momentum:

$$r v_\theta + \frac{1}{2}fr^2 = \mathrm{const.}$$

Note we have negelected the variation of f. We are making a special planar approximation: *the f-plane*.

A powerful equation!

$$rv + \frac{1}{2}fr^2 = \mathrm{const.}$$

(Note: $v_\theta \to v$). Consider a ring of air with initial radius $r = 200$ km and $v_\theta = 0$. The ring is at low latitudes where $f = 0.5 \times 10^{-4}\ \mathrm{s}^{-1}$. The ring contracts to a radius of $r = 100$ km. What is the resulting value of v_θ?

$$r_2 v_2 + \frac{1}{2}fr_2^2 = r_1 v_1 + \frac{1}{2}fr_1^2$$

$$v_2 = \frac{1}{2}f\frac{r_1^2 - r_2^2}{r_2}$$

$$v_2 = 7.5\ \mathrm{m\ s}^{-1}$$