METR 5223: Atmospheric Radiation

Elementary Adjustment to Radiative Equilibrium

Lecture for Spring 2009

Prof. Brian H. Fiedler

School of Meteorology, University of Oklahoma

$$C\frac{dT}{dt} = s - \sigma T^4$$

- C is the heat capacity per unit area, *e. g.* $\rho H c_w$ for a layer of water of depth H.
- s is the absorbed radiation, *e. g.* $\frac{1}{4}S_0(1 - \alpha_p)$
- σT^4 assumes perfect black body emission

We have studied the equilibrium case:

$$0 = s - \sigma T^4$$

Question: Using $s = \sigma T^4$, if s increases by 1%, by what percent does T increase?

Let:

$$T(t) = \overline{T} + T'(t)$$

$$s(t) = \overline{s} + s'(t)$$

$\overline{T}$ will be defined in terms of $\overline{s}$.

We will assume $T' << \overline{T}$

$$T^4 = (\overline{T} + T')^4 = \overline{T}^4 + 4\overline{T}^3 T' + 6\overline{T}^2 T'^2 + 4\overline{T}^3 T' + T'^4$$

$$T^4 \approx \overline{T}^4 + 4\overline{T}^3 T'$$

$$s - \sigma T^4 = \overline{s} + s' - \sigma\overline{T}^4 - \sigma 4\overline{T}^3 T'$$

$$\overline{s} - \sigma\overline{T}^4 = 0$$

$$C\frac{dT'}{dt} = \overline{s} + s' - \sigma\overline{T}^4 - 4\sigma\overline{T}^3 T'$$

$T'(t)$ is the fluctuation or departure of T from the equilibrium value $\overline{T}$.

$$\tau\frac{dT'}{dt} + T' = T'_0$$

$$\tau \equiv \frac{C}{4\sigma\overline{T}^3}$$

$$T'_0(t) \equiv \frac{s'}{4\sigma\overline{T}^3}$$

For $\tau = 0$, $T'(t) = T'_0(t)$.

T'_0 is the *no-lag response* to the changing $s'(t)$.

$$\tau \frac{dT'}{dt} + T' = T'_0$$

Suppose $T'_0 = 0$:

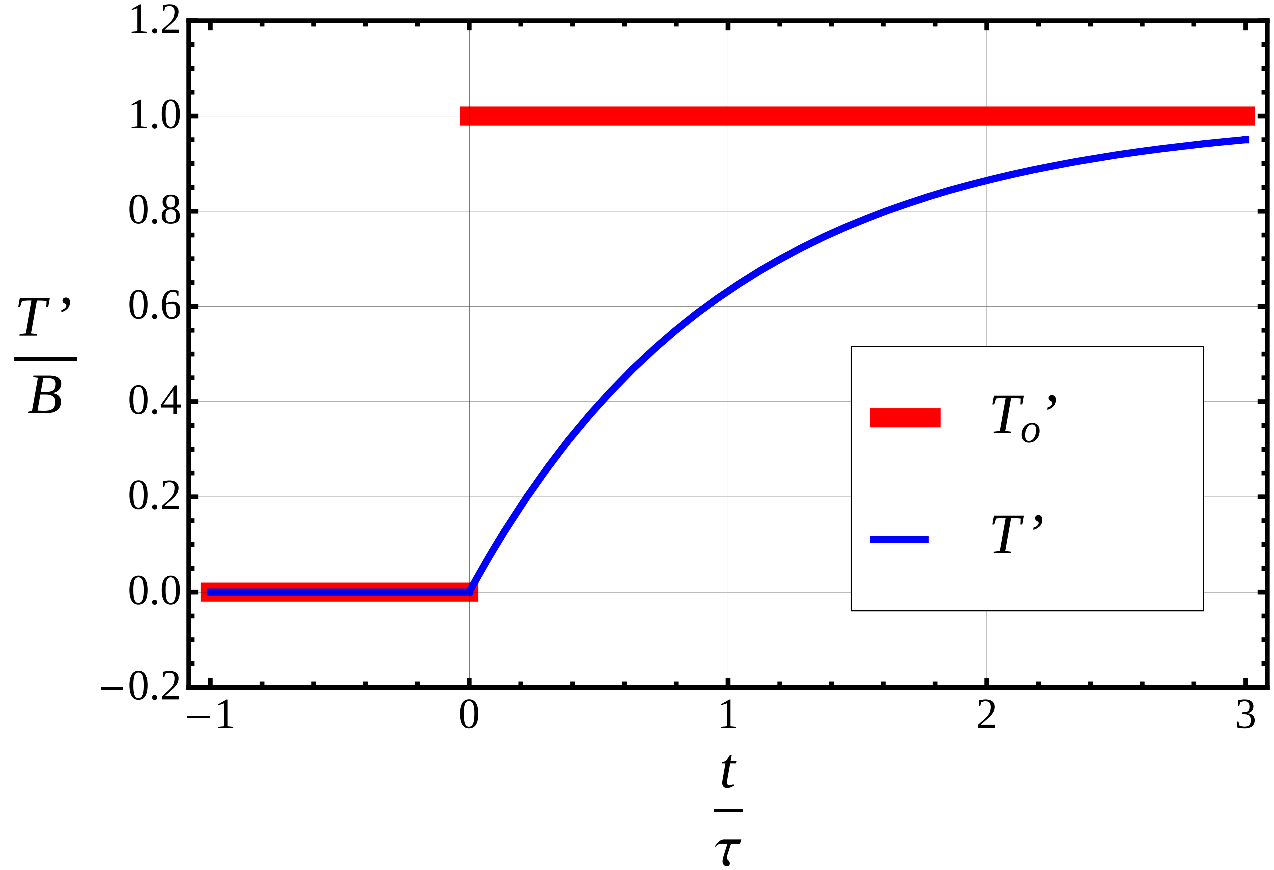
$$T'(t) = Ae^{-t/\tau}$$

Suppose $s' = 0$ for $t < 0$ and $s' =$ constant for $t \geq 0$: So we consider $T'_0 = 0$ for $t < 0$ and $T'_0 = B$ for $t > 0$. We assume $T'(0) = 0$. For $t > 0$:

$$T'(t) = Ae^{-t/\tau} + B$$

With $T'(0) = 0$:

$$T'(t) = B\left(1 - e^{-t/\tau}\right)$$



$$T'(t) = B\left(1 - e^{-t/\tau}\right)$$

If Earth's oceans were distributed over the entire globe, the depth would be $H = 2738$ m. With

$$\tau \equiv \frac{C}{4\sigma\overline{T}^3}$$

and

$$C = \rho H c_w,$$

$$\tau = 68 \text{ years} \qquad (\overline{T} = 288K)$$

$$\tau = 97 \text{ years} \qquad (\overline{T} = 255K)$$

A very important number in global warming predictions.

Instead of T_0' changing instantly at $t = 0$, consider

$$T_0' = Be^{t/\tau_r}$$

τ_r is the time scale for the "ramp-up" of T_0'.

Explore how the solution depends on the ratio of the two time scales, τ/τ_r.

$$\tau \frac{dT'}{dt} + T' = Be^{t/\tau_r}$$

Propose

$$T'(t) = Ae^{t/\tau_r}$$

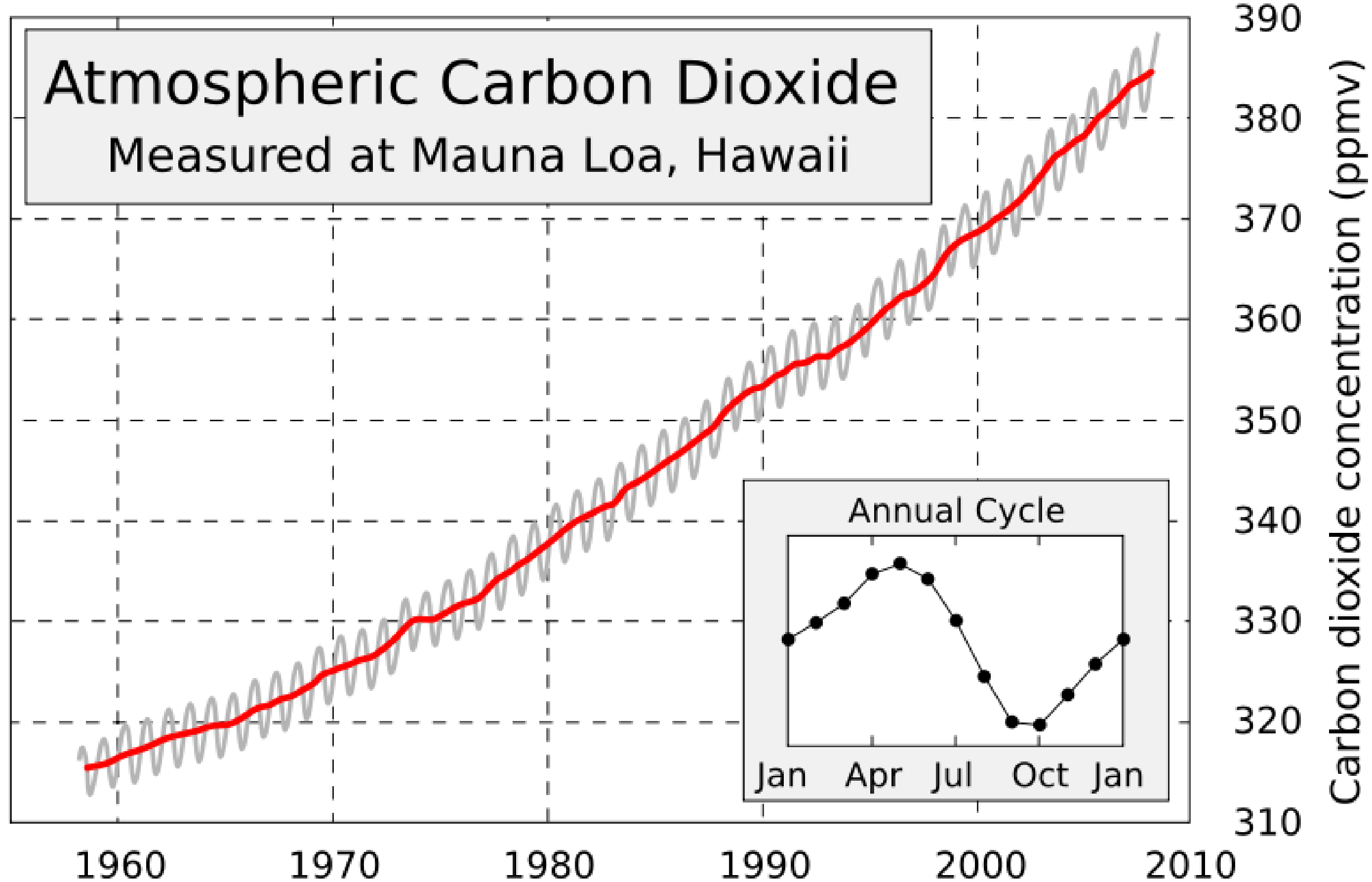
$$\tau \frac{d}{dt} A e^{t/\tau_r} + A e^{t/\tau_r} = B e^{t/\tau_r}$$

$$\frac{\tau}{\tau_r} A e^{t/\tau_r} + A e^{t/\tau_r} = B e^{t/\tau_r}$$

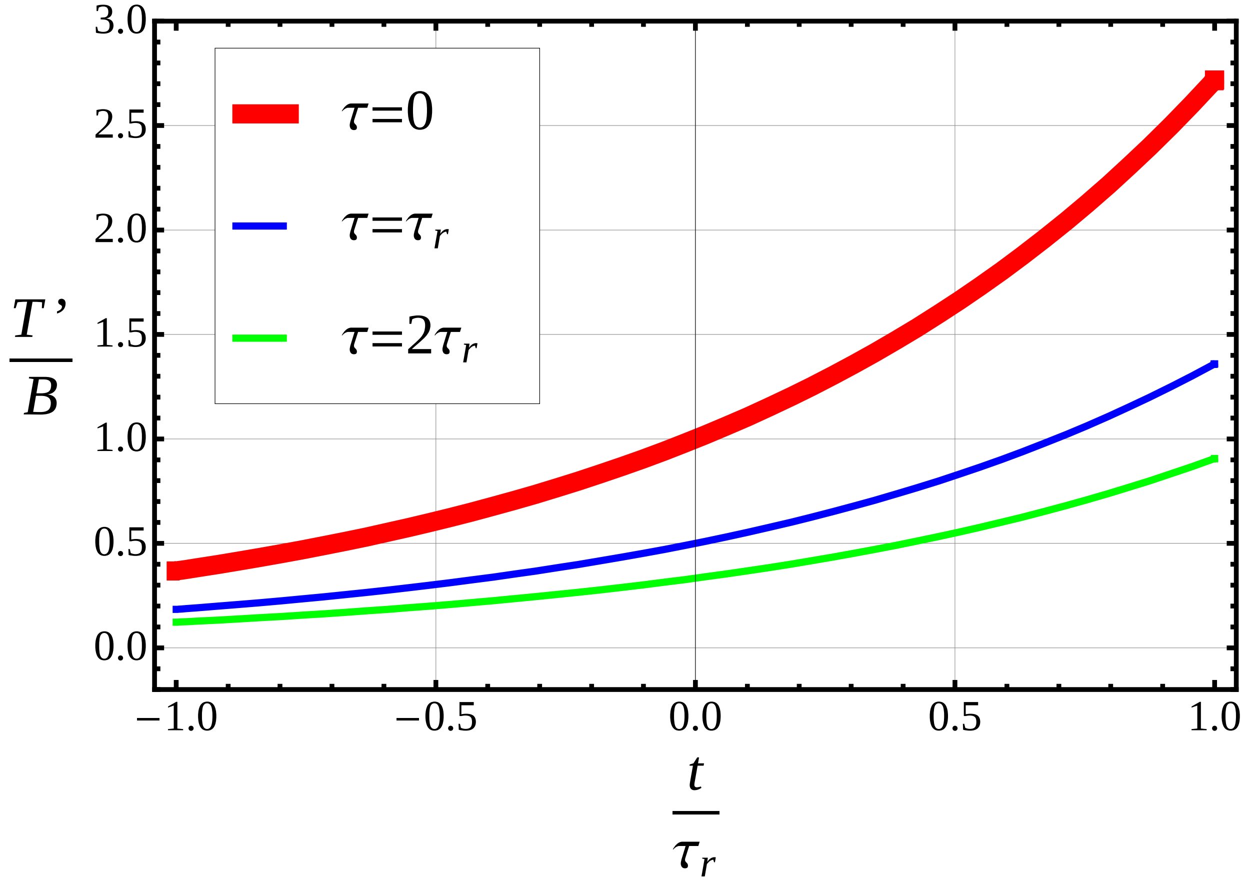
$$A \left(1 + \frac{\tau}{\tau_r} \right) = B$$

$$\frac{A}{B} = \frac{1}{1 + \frac{\tau}{\tau_r}}$$

The amplitude is diminished for larger τ/τ_r.



The increase of CO_2 from the preindustrial value of 270 ppmv is not exactly exponential. But it appears to be approximately exponential with e-folding time of about 50 years. Using 50 years for τ_r gives $A/B \approx \frac{1}{2}$.



$$T'(t) = \frac{1}{1+\frac{\tau}{\tau_r}} B e^{-t/\tau_r}$$

Next consider seasonal forcing,

$$T_0' = B\cos(\omega t)$$

$$\tau\frac{dT'}{dt} + T' = B\cos(\omega t)$$

To aid in comparison in the effect of the time scales, we let τ_o be the oscillation time scale, the time to go through one radian (not one cycle) so:

$$\omega = \frac{1}{\tau_o}$$

Propose

$$T'(t) = A\cos(\omega t - \delta)$$

$$\frac{dT'}{dt} = -A\omega\sin(\omega t - \delta)$$

$$-A\omega\tau\sin(\omega t-\delta)+A\cos(\omega t-\delta)=B\cos(\omega t)$$

$$\begin{aligned} - &\quad A\omega\tau\left[\sin(\omega t)\cos(\delta)-\cos(\omega t)\sin(\delta)\right] \\ + &\quad A\left[\cos(\omega t)\cos(\delta)+\sin(\omega t)\sin(\delta)\right]=B\cos(\omega t) \end{aligned}$$

What is the next step?

$$A \quad \cos(\omega t)\,[\cos(\delta) + \omega\tau \sin(\delta)]$$

$$A \quad \sin(\omega t)[-\omega\tau \cos(\delta) + \sin(\delta)] = B\cos(\omega t)$$

The only possibility for equality for all values of t and all values of $\cos(\omega t)$ and $\sin(\omega t)$ is:

$$[-\omega\tau \cos(\delta) + \sin(\delta)] = 0$$

or

$$\tan(\delta) = \omega\tau \qquad \delta = \tan^{-1}(\omega\tau)$$

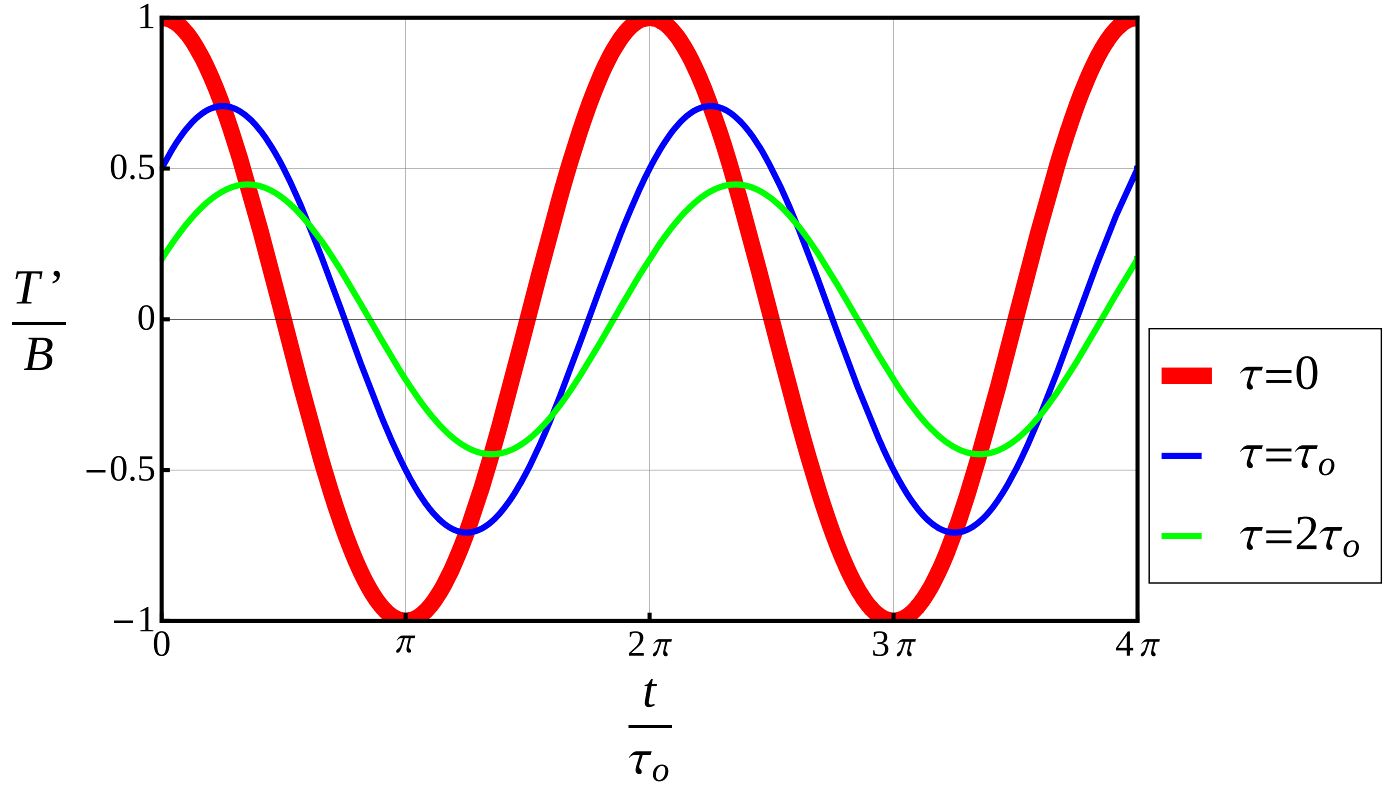
For small $\omega\tau$, $\delta \to 0$ and the response T' is in phase with the forcing T'_0. For large $\omega\tau$, $\delta \to 90°$, and the response lags the forcing by a quarter cycle.

The remaining part of the equation is:

$$A\cos(\omega t)\left[\cos(\delta)+\omega\tau\sin(\delta)\right]=B\cos(\omega t)$$

$$A\left[\cos(\delta)+\tan(\delta)\sin(\delta)\right]=B$$

$$\frac{A}{B}=\cos(\delta)=\frac{1}{\sqrt{1+\omega^2\tau^2}}=\frac{1}{\sqrt{1+\frac{\tau^2}{\tau_o^2}}}$$



$$T'(t) = \frac{1}{\sqrt{1+\frac{\tau^2}{\tau_o^2}}} B \cos\left(\frac{t}{\tau_o} - \delta\right) \qquad \delta = \tan^{-1}\left(\frac{\tau}{\tau_o}\right)$$

For a seasonal cycle

$$\tau_o = \frac{1 \;\; \mathrm{yr}}{2\pi}$$

An oceanic mixed layer of depth $H = 20$ m, gives $\tau = 0.49$ yr:

$$\frac{\tau}{\tau_o} = 3.1$$

$$\delta = 72^\circ$$

$$A/B = \cos(\delta) = 0.3$$

In a previous lecture we studied a simple model for radiative equilibrium with a simple greenhouse. The equilibrium condition for the atmospheric slab was:

$$0 = \epsilon\sigma T_s^4 - 2\epsilon\sigma T_a^4$$

This more general energy equation for the slab would be

$$C_a \frac{dT_a}{dt} = \epsilon\sigma T_s^4 - 2\epsilon\sigma T_a^4$$

Suppose T_a is not in equilibrium, but T_s is very near equilibrium. Let

$$T_a = \overline{T}_a + T_a'(t) \qquad T_s = \overline{T}_s$$

$$C_a \frac{dT_a'}{dt} = \epsilon\sigma \overline{T}_s^4 - 2\epsilon\sigma \left(\overline{T}_a^4 + 4\overline{T}_a^3 T_a' \right)$$

The equilibrium solution has the property:

$$\sigma \overline{T}_s^4 = 2\sigma \overline{T}_a^4$$

The adjustment equation simplifies to:

$$C_a \frac{dT'_a}{dt} = -8\epsilon\sigma \overline{T}_a^3 T'_a$$

The adjustment time τ is calculated with:

$$C_a = \frac{p_0}{g} c_p$$

$$\tau = \frac{C_a}{8\epsilon\sigma \overline{T}_a^3} = \frac{p_0 c_p}{g 8\epsilon\sigma \overline{T}_a^3} = 19 \text{ days}$$