METR 5223: Atmospheric Radiation

Integration of the solution to the Schwarzschild equation over ν

Lecture for Spring 2009, v0.3

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Schwarzschild equation:

$$\frac{dL_\nu}{ds} = -k_\nu \rho_a L_\nu + k_\nu \rho_a B_\nu$$

Define *optical path* χ_ν:

$$\chi_\nu(s) = \int_{s_0}^{s} k_\nu(s')\rho_a(s')ds'$$

or sometimes, in the case of constant k_ν:

$$\chi_\nu(s) = k_\nu \int_{s_0}^{s} \rho_a(s')ds' \equiv k_\nu u$$

So

$$\frac{dL_\nu}{d\chi_\nu} = -L_\nu + B_\nu$$

The solution is:

$$L_\nu(\chi_\nu) = L_\nu(0)e^{-\chi_\nu} + \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu)e^{-(\chi_\nu-\chi'_\nu)}d\chi'_\nu$$

Also applies to F_ν with slight modifications:

$$F_\nu(\chi_\nu) = F_\nu(0)e^{-\chi_\nu} + \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} \pi B_\nu(\chi'_\nu)e^{-(\chi_\nu-\chi'_\nu)}d\chi'_\nu$$

and with the *diffuse approximation*:

$$u \longrightarrow \frac{5}{3}u$$

Let the *transmittance* be:

$$\tau_\nu(\chi'_\nu, \chi_\nu) \equiv e^{-(\chi_\nu - \chi'_\nu)}$$

The solution can be written:

$$F_\nu(\chi_\nu) = F_\nu(0)\tau_\nu(0, \chi_\nu) + \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} \pi B_\nu(\chi'_\nu)\tau(\chi'_\nu, \chi_\nu) d\chi'_\nu$$

But note for monochromatic radiation:

$$\frac{d\tau_\nu}{d\chi'_\nu} = \tau_\nu(\chi'_\nu, \chi_\nu)$$

The solution can be written:

$$F_\nu(\chi_\nu) = F_\nu(0)\tau_\nu(0,\chi_\nu) + \int\limits_{\tau'_\nu=\tau_\nu(0,\chi_\nu)}^{\tau'_\nu=1} \pi B_\nu(T) d\tau'_\nu$$

Check case of constant T:

$$F_\nu(\chi_\nu) = F_\nu(0)\tau_\nu(0,\chi_\nu) + \pi B_\nu(T)(1 - \tau_\nu(0,\chi_\nu))$$

Contemplate numerical solution:

$$\begin{aligned} F_\nu(\chi_\nu) = \quad & F_\nu(0)\tau_\nu(0,\chi_\nu) + \\ & \sum_{n=1}^{N} \pi B_\nu(T_n) \left[\tau_\nu(\chi_{\nu,n+},\chi_\nu) - \tau_\nu(\chi_{\nu,n-},\chi_\nu)\right] \end{aligned}$$

Integration over frequency ν. Let

$$F_i = \int_{\nu=\nu_{1i}}^{\nu=\nu_{2i}} F_\nu d\nu \equiv \int_i F_\nu d\nu$$

Consider the downward flux at the surface, using normalized pressure for the integral:

$$F_i = \int_i \int_{p=0}^{p=1} \pi B_\nu(T) \frac{d\tau_\nu}{dp} dp d\nu$$

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Assume $B_\nu(T) = \frac{1}{\Delta\nu_i} B_i(T)$, because frequency interval is narrow and B_ν doesn't vary much in it:

$$F_i = \int_{p=0}^{p=1} \pi \frac{1}{\Delta\nu_i} B_i(T) \int_i \frac{d\tau_\nu}{dp} d\nu dp$$

$$F_i = \int_{p=0}^{p=1} \pi \frac{1}{\Delta\nu_i} B_i(T) \frac{d}{dp} \int_i \tau_\nu d\nu dp$$

$$F_i = \int_{p=0}^{p=1} \pi \frac{1}{\Delta\nu_i} B_i(T) \frac{d}{dp} (\Delta\nu_i \overline{\tau}_i) dp$$

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Contemplate numerical solution:

$$F_i = \sum_{n=1}^{N} \pi B_{i,n} \Delta\overline{\tau}_{i,n}$$

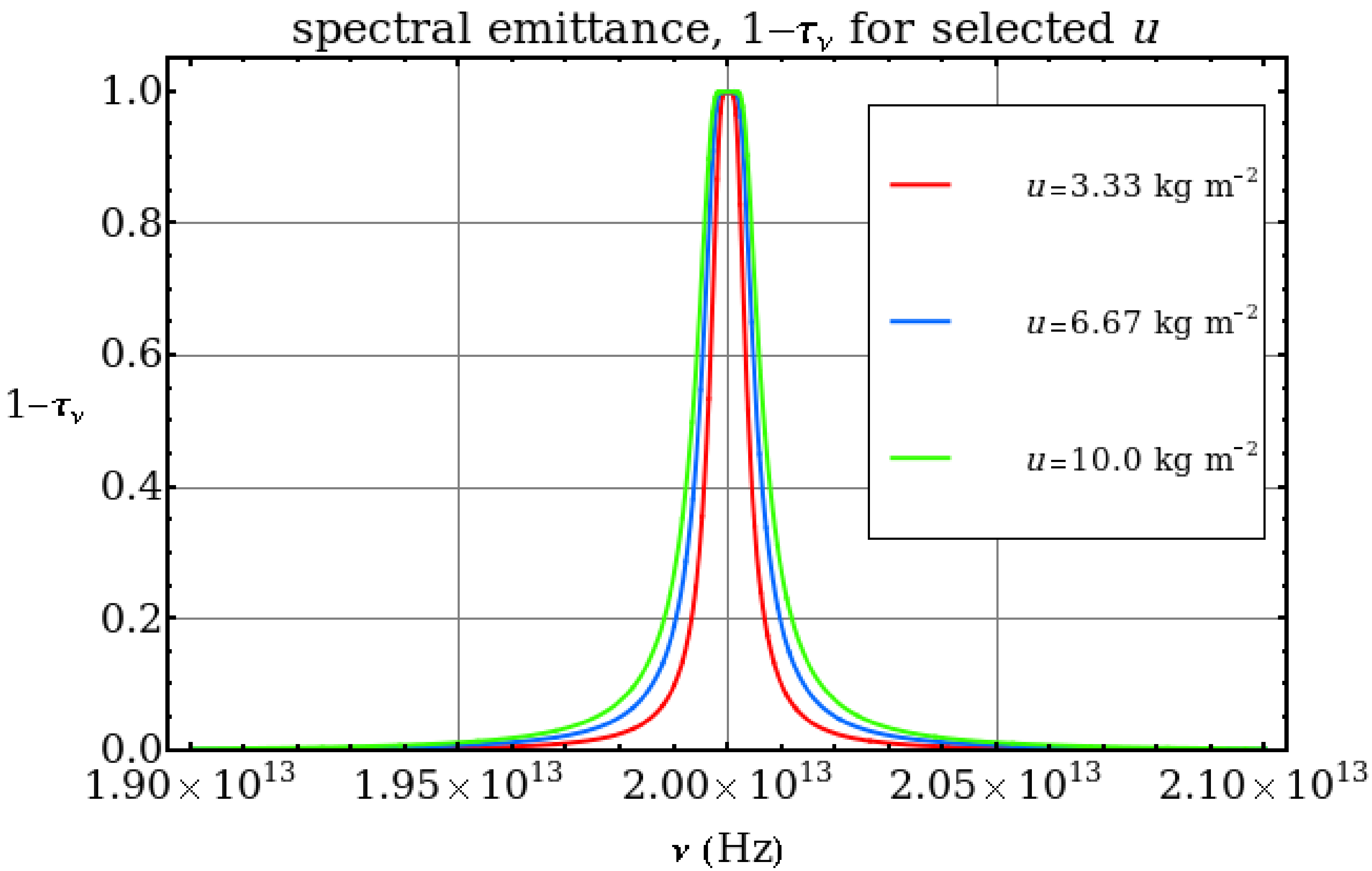
Except for idealized band models (for example, the Lorentz profile), obtaining $\Delta\overline{\tau}_{i,n}$ requires much numerical computation with empirical data.

$$\overline{\epsilon}_{i,n} = 1 - \Delta\overline{\tau}_{i,n}$$

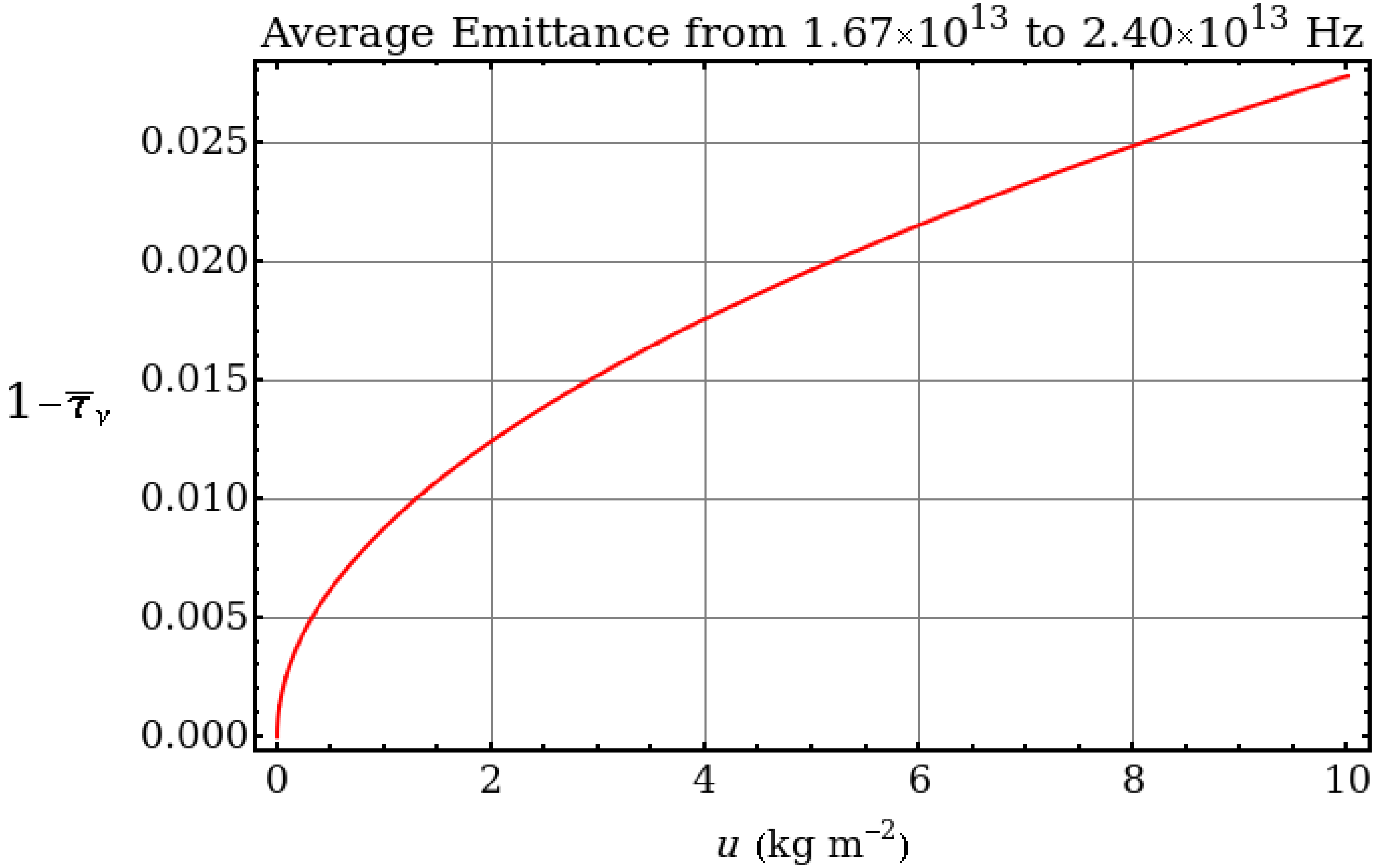
Except for the sign change working with $\Delta\overline{\epsilon}_{i,n}$ is the same as working with $\Delta\overline{\tau}_{i,n}$.

Employing the average emissivity $\overline{\epsilon}_i$ across a portion of the Planck spectrum is is traditionally called a *broadband emissivity model* (or sometimes *wide-band*).

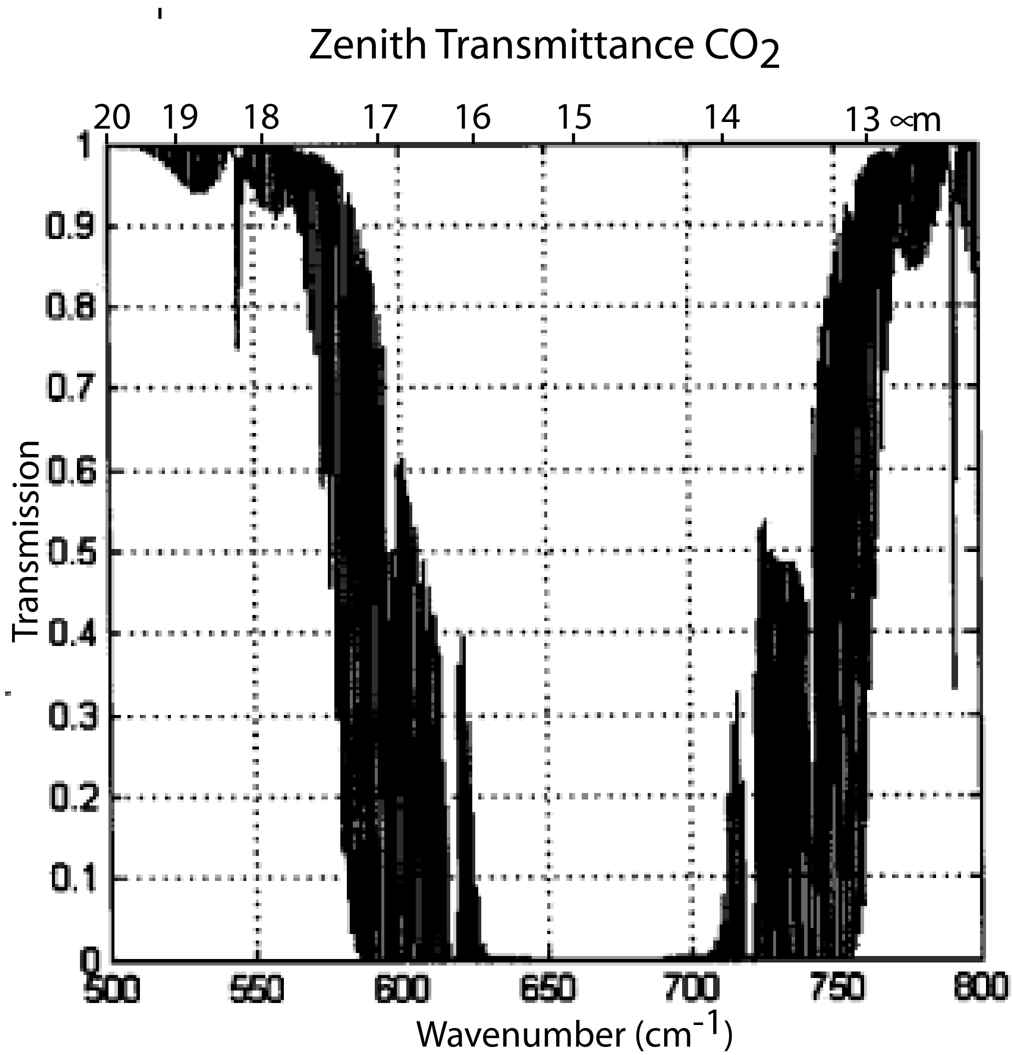
Warning: if a single layer has transmittance $\overline{\tau}_i$, the transmittance for two layers is NOT $\overline{\tau}_i^2$, as it was for monochromatic radiation.



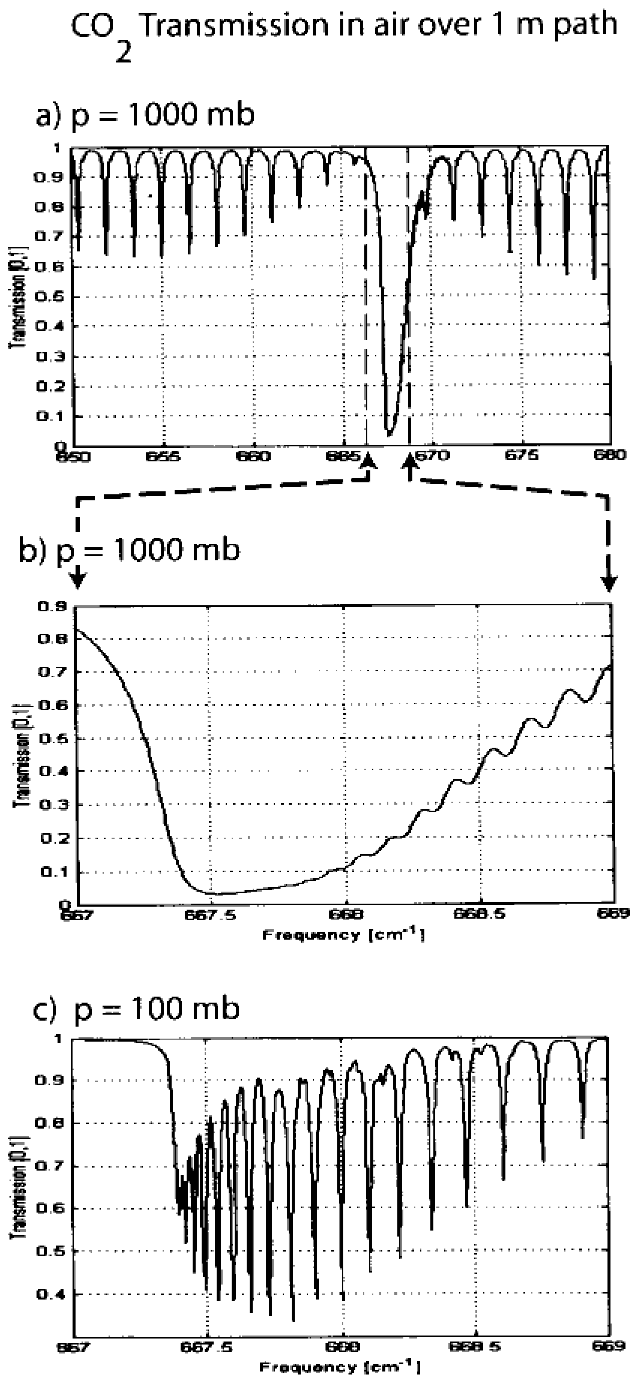
Emittance example for Lorentz profile with $S = 1.0 \times 10^{11}$ Hz m^2 kg^{-1}, $\nu_0 = 2.0 \times 10^{13}$ Hz, $\gamma = .001 \times 10^{13}$ Hz, and total mass path $U_\circ = 10$ kg m^2.



For this Lorentz profile, the optically thick limit applies, with equivalent width $W = 2\sqrt{Su\gamma}$. The average emittance values are easily calculated, or obtained from the graph.



Zenith transmittance of the atmosphere due to CO_2 in the vicinity of 15 μm.



High resolution depiction of the transmission spectrum of a one-meter path through air with typical CO_2 concentration.

FYI:

$$F_i = \int_{p=0}^{p=1} \pi B_i(T) \frac{d}{dp} \overline{\tau}_i dp$$

can be transformed using "integration by parts" to

$$F_i = \int_{p=0}^{p=1} \frac{d}{dp} \left[\pi B_i(T) \overline{\tau}_i\right] dp - \int_{p=0}^{p=1} \overline{\tau}_i \frac{d}{dp} \left[\pi B_i(T)\right] dp$$

$$F_i = \pi B_i(T) \overline{\tau}_i \Big|_{p=0}^{p=1} - \int_{p=0}^{p=1} \overline{\tau}_i \frac{d}{dp} \left[\pi B_i(T)\right] dp$$

Such forms are used in Roach and Slingo (1979).