METR 5223: Atmospheric Radiation

Solution to the Schwarzschild equation for IR

Lecture for Spring 2009

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Consider an atmosphere of *constant temperature* T_a and a black-body surface with temperature T_s. The solution for the upward *spectral radiance* is:

$$L_\nu(\chi_\nu) = B_\nu(T_a)\left(1 - e^{-\chi_\nu}\right) + B_\nu(T_s)e^{-\chi_\nu}$$

where the *optical path* χ_ν is:

$$\chi_\nu(s) = \int_{s_0}^{s} k_\nu(s')\rho_a(s')ds'$$

Similarly,

$$\nu L_\nu(\chi_\nu) = \nu B_\nu(T_a)\left(1 - e^{-\chi_\nu}\right) + \nu B_\nu(T_s)e^{-\chi_\nu}$$

In a laboratory, with constant pressure along a tube, k_ν could be indepedent of position. We could write

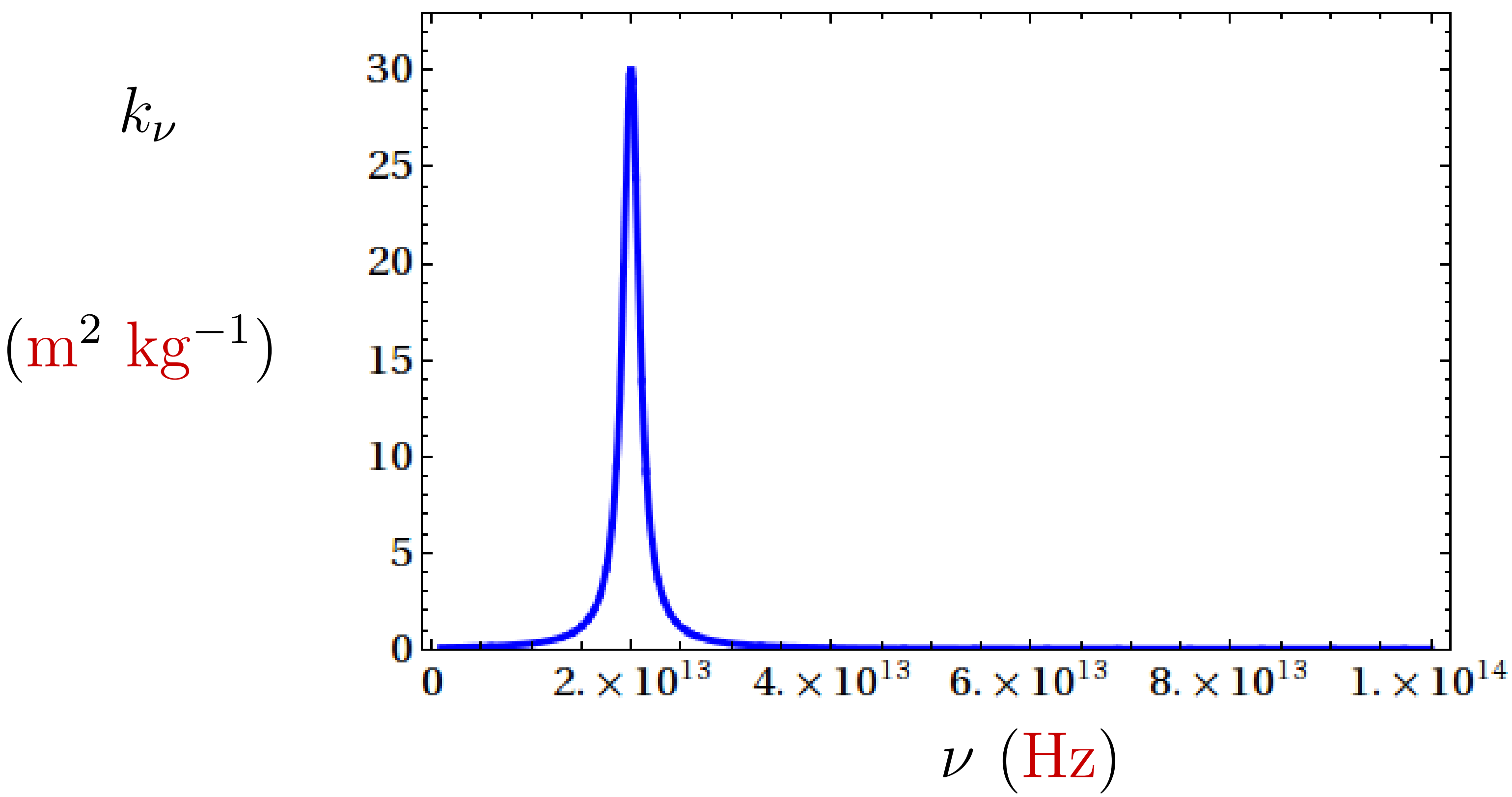
$$\chi_\nu(s) = k_\nu \int\limits_{s_0}^{s} \rho_a(s')ds' \equiv k_\nu u$$

where u is the *mass path*. A Lorentz profile for $k_n u$ could be written as:

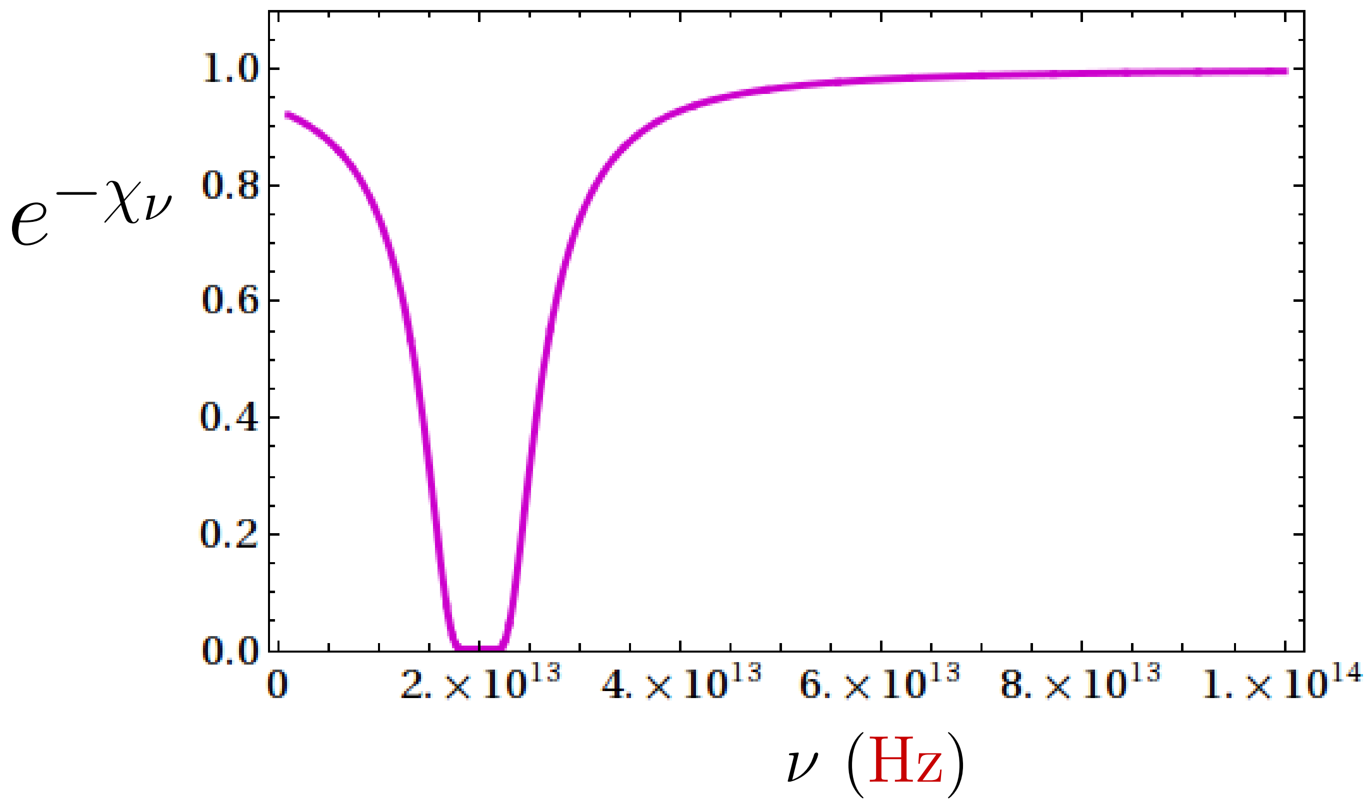
$$k_\nu = k_{\max} \frac{\gamma^2}{\gamma^2 + (\nu - \nu_0)^2}$$

Example k_ν with

$k_{\max} = 30$ m^2 kg^{-1}, $\gamma = 0.1 \times 10^{13}$ Hz and $\nu_0 = 2 \times 10^{13}$ Hz.

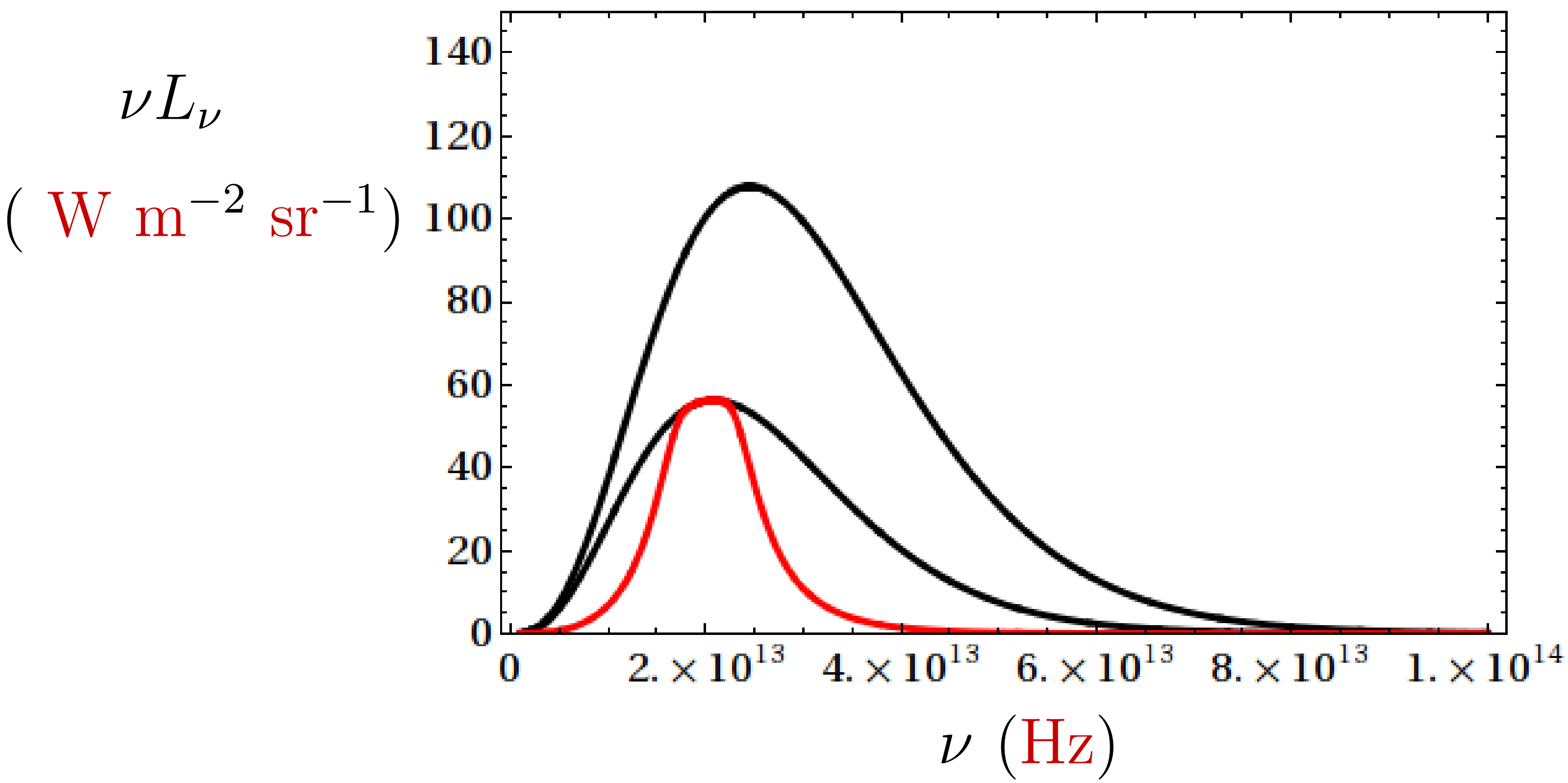


Continue the example, take $u = 1.0$ kg m^{-2}. With $\chi_\nu = k_\nu u$, the maximum in χ_ν is 30. Here the *transmittance* is $e^{-\chi_\nu} = e^{-k_\nu u}$.



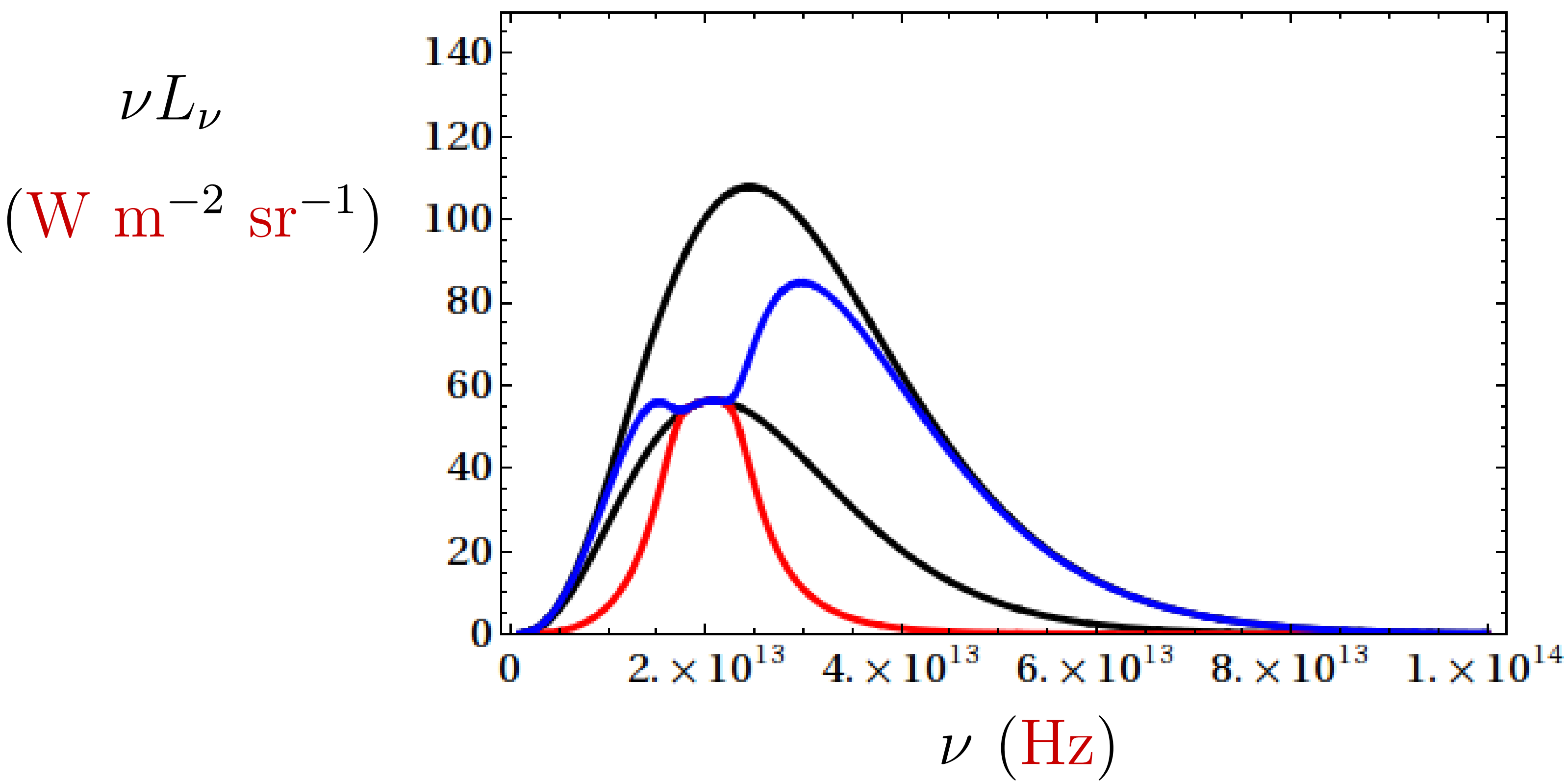
Continue the example with $T_s = 300$ K and $T_a = 255$ K. Red is spectral radiance down to the surface, or

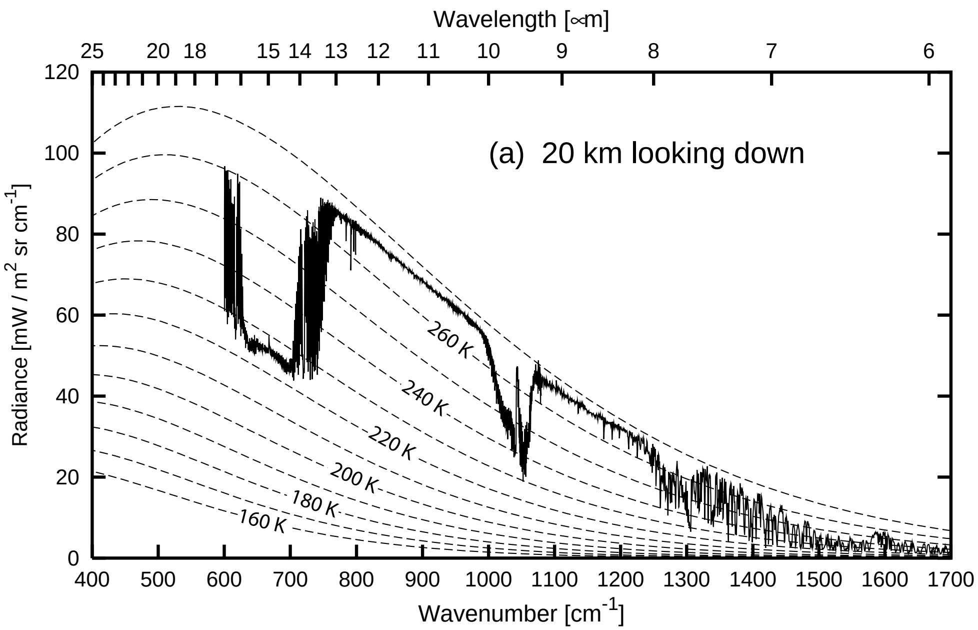
$$\left(1 - e^{-\chi_\nu}\right) \nu B_\nu(T_a)$$



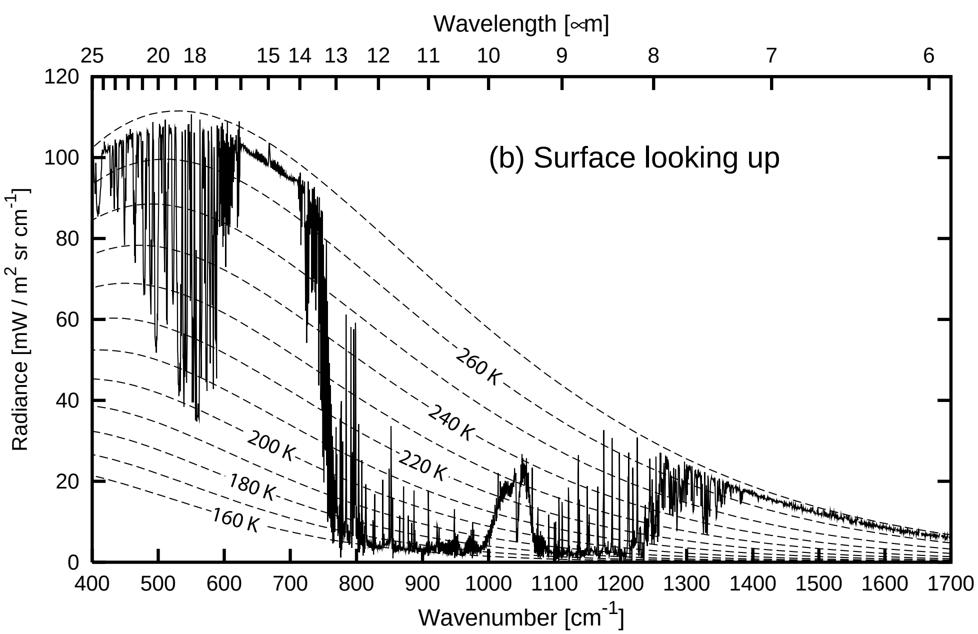
Blue is spectral radiance out of the top of atmosphere,

$$\left(1 - e^{-\chi_\nu}\right) \nu B_\nu(T_a) + e^{-\chi_\nu} \nu B_\nu(T_s)$$





Looking down through cloud-free atmosphere at polar ice sheet.



Looking up at cloud-free atmosphere from polar ice sheet.

Now consider the solution to the Schwarzschild equation with **non-constant** T **and** B_ν :

$$L_\nu(\chi_\nu) = L_\nu(0)e^{-\chi_\nu} + \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu)e^{-(\chi_\nu-\chi'_\nu)}d\chi'_\nu$$

Consider solution at $\chi_{\nu\mathrm{m}}$, maximum χ_ν for traversal of the whole atmosphere. Also seek solution "per $\ln\nu$" rather than "per ν "

$$\nu L_\nu(\chi_{\nu\mathrm{m}}) = \nu L_\nu(0)e^{-\chi_{\nu\mathrm{m}}} + \int_{\chi'_\nu=0}^{\chi'_\nu=\chi_{\nu\mathrm{m}}} \nu B_\nu(\chi'_\nu)e^{-(\chi_{\nu\mathrm{m}}-\chi'_\nu)}d\chi'_\nu$$

- Let p be normalized pressure, varying from 1 at the surface to 0 at the top of the atmosphere.
- Assume $T(p)$ is known.
- For upward travel: $\chi'_\nu = \chi_{\nu\mathrm{m}}(1-p)$
- For downward travel: $\chi'_\nu = \chi_{\nu\mathrm{m}}\, p$

For upward travel:

$$\chi'_\nu = \chi_{\nu\mathrm{m}}(1-p) \qquad d\chi'_\nu = -\chi_{\nu\mathrm{m}}\, dp \qquad \chi_{\nu\mathrm{m}} - \chi'_\nu = \chi_{\nu\mathrm{m}}\, p$$

$$\nu L_\nu(\chi_{\nu\mathrm{m}}) = \nu L_\nu(0) e^{-\chi_{\nu\mathrm{m}}} + \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_{\nu\mathrm{m}}} \nu B_\nu(\chi'_\nu) e^{-(\chi_{\nu\mathrm{m}}-\chi'_\nu)} d\chi'_\nu$$

$$\nu L_\nu(\chi_{\nu\mathrm{m}}) = \nu B_\nu(T(1)) e^{-\chi_{\nu\mathrm{m}}} - \chi_{\nu\mathrm{m}} \int\limits_{p=1}^{p=0} \nu B_\nu(T(p)) e^{-\chi_{\nu\mathrm{m}}\, p} dp$$

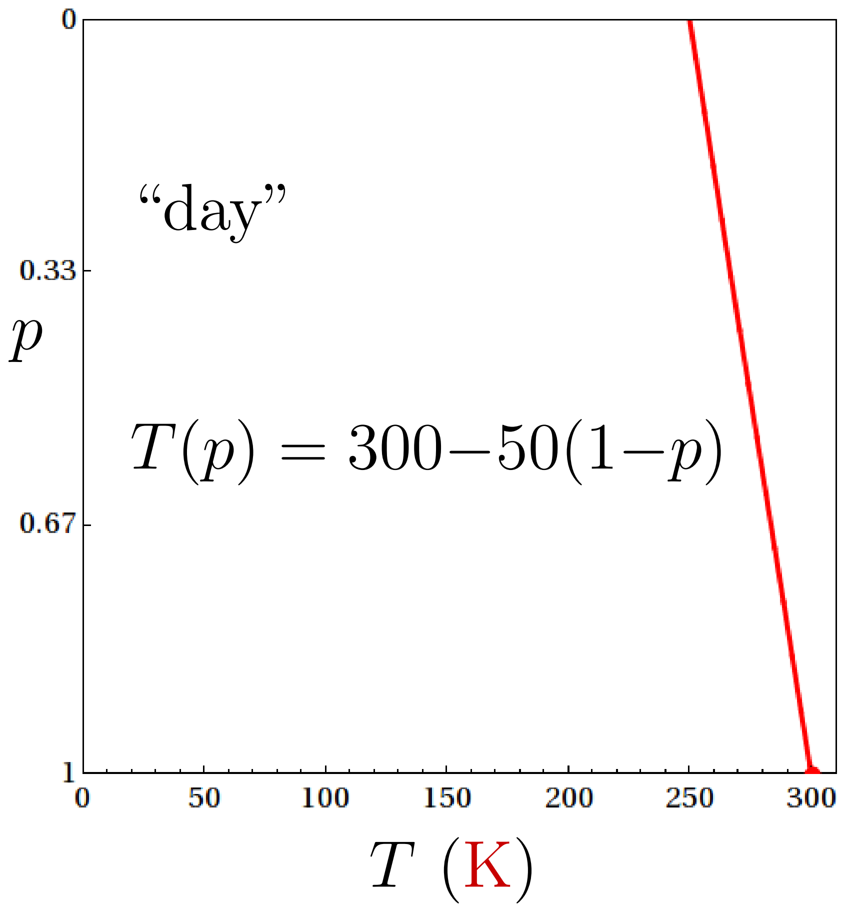
For downward travel:

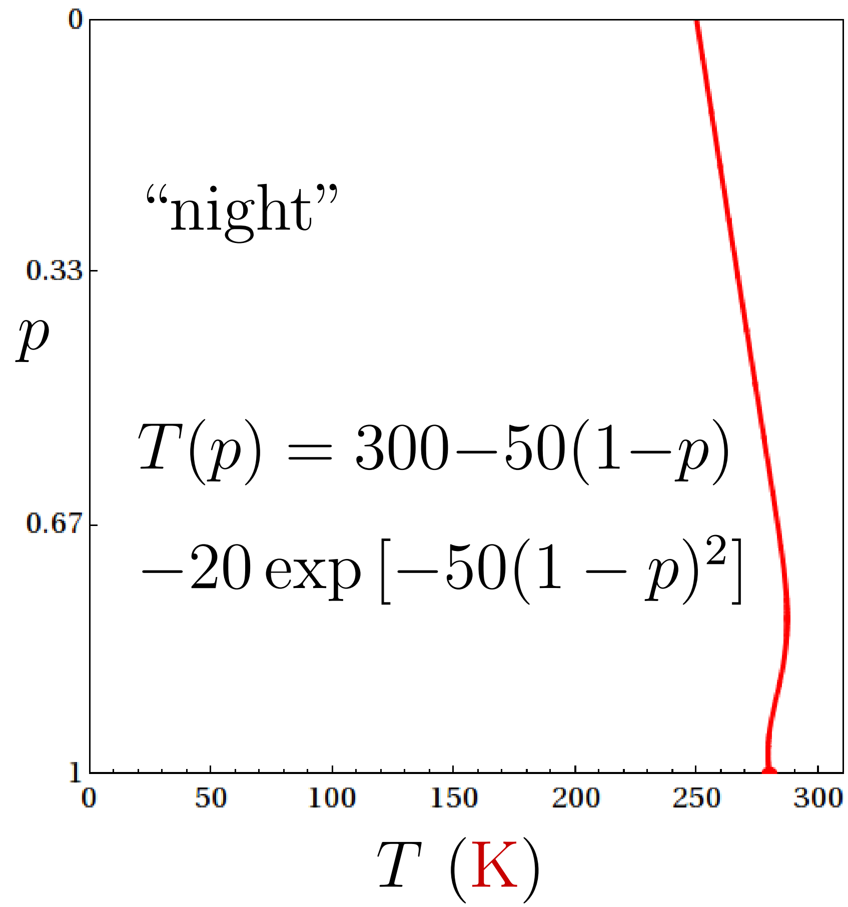
$$\chi'_\nu = \chi_{\nu\mathrm{m}} p \qquad d\chi'_\nu = \chi_{\nu\mathrm{m}}\, dp \qquad \chi_{\nu\mathrm{m}} - \chi'_\nu = \chi_{\nu\mathrm{m}}(1-p)$$

$$\nu L_\nu(\chi_{\nu\mathrm{m}}) = \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_{\nu\mathrm{m}}} \nu B_\nu(\chi'_\nu) e^{-(\chi_{\nu\mathrm{m}}-\chi'_\nu)} d\chi'_\nu$$

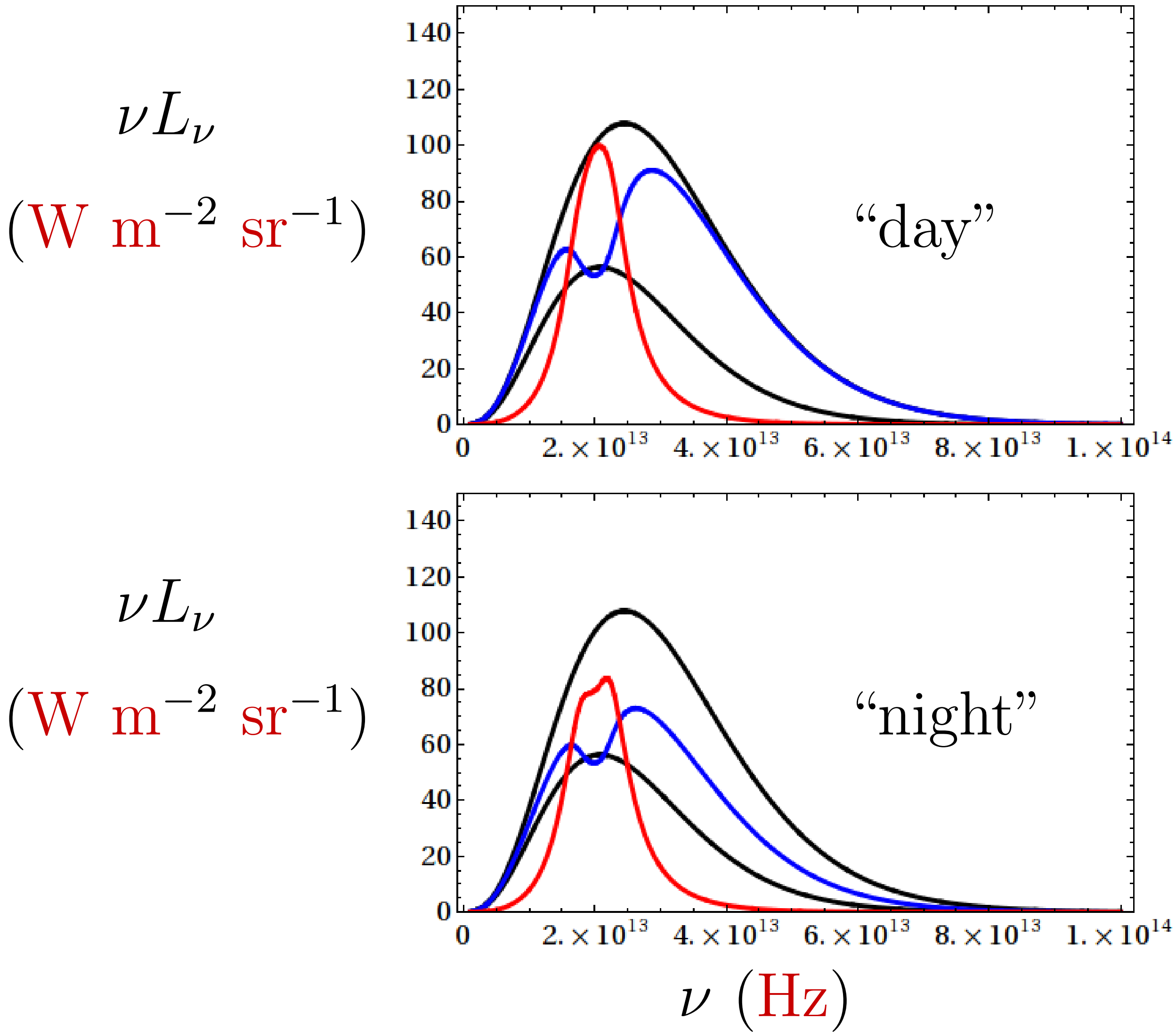
$$\nu L_\nu(\chi_{\nu\mathrm{m}}) = \chi_{\nu\mathrm{m}} \int\limits_{p=0}^{p=1} \nu B_\nu(T(p)) e^{-\chi_{\nu\mathrm{m}}(1-p)} dp$$

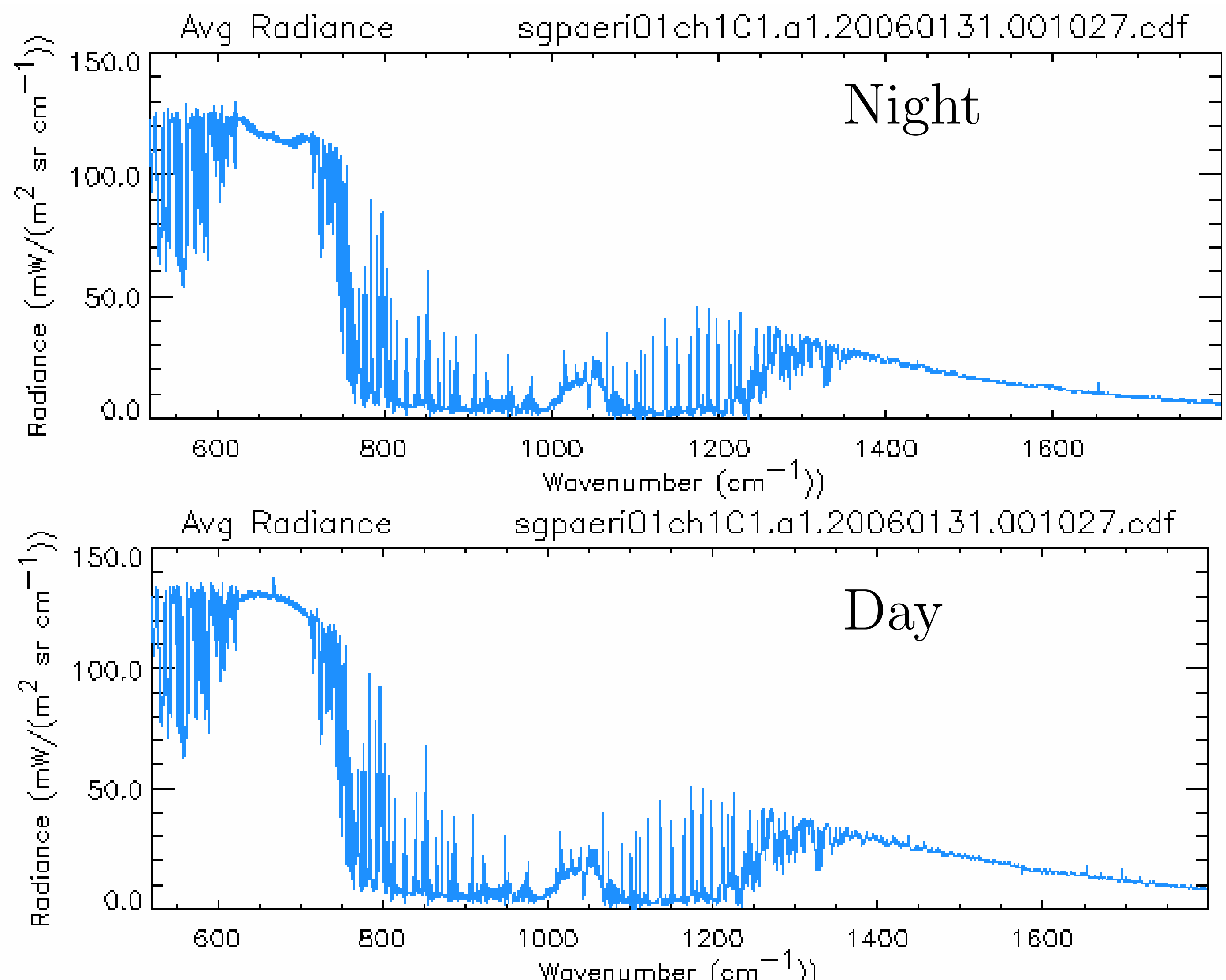
Consider some simple profiles for $T(p)$:





Solutions for $L_\nu(\nu)$:





Measurements of $L_\nu(\nu)$ at an ARM site. The cool boundary layer reduces the spectral radiance in the center of the optically thick part of the CO_2 band.

Recap: the Schwarzschild equation is:

$$\frac{dL_\nu}{d\chi_\nu} = -L_\nu + B_\nu$$

The solution is:

$$L_\nu(\chi_\nu) = L_\nu(0)e^{-\chi_\nu} + \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu)e^{-(\chi_\nu-\chi'_\nu)}d\chi'_\nu$$

with "optical path" defined:

$$\chi_\nu(s) = \int\limits_{s_0}^{s} k_\nu(s')\rho_a(s')ds'$$

Using the transmittance in the solution

Let the *transmittance* be:

$$\tau_\nu(\chi'_\nu, \chi_\nu) \equiv e^{-(\chi_\nu - \chi'_\nu)} = e^{\chi'_\nu - \chi_\nu}$$

The solution can be written:

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) + \int\limits_{\chi'_\nu=0}^{\chi'_\nu=\chi_\nu} B_\nu(\chi'_\nu)\tau(\chi'_\nu, \chi_\nu)d\chi'_\nu$$

But note:

$$\frac{d\tau_\nu}{d\chi'_\nu} = \frac{d}{d\chi'_\nu} e^{\chi'_\nu - \chi_\nu} = e^{\chi'_\nu - \chi_\nu} = \tau_\nu(\chi'_\nu, \chi_\nu)$$

The solution can be written:

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0,\chi_\nu) + \int\limits_{\tau'_\nu = \tau_\nu(0,\chi_\nu)}^{\tau'_\nu = 1} B_\nu(T)\, d\tau'_\nu$$

Check case of constant T:

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0,\chi_\nu) + B_\nu(T)(1 - \tau_\nu(0,\chi_\nu))$$

Contemplate numerical solution:

$$\begin{aligned} L_\nu(\chi_\nu) = \quad & L_\nu(0)\tau_\nu(0,\chi_\nu) + \\ & \sum_{n=1}^{N} B_\nu(T_n)\left[\tau_\nu(\chi_{\nu,n+},\chi_\nu) - \tau_\nu(\chi_{\nu,n-},\chi_\nu)\right] \end{aligned}$$

Where T_n is the temperature of layer n, $\chi_{\nu,n+}$ is the optical path for the side of layer nearest the detector, and $\chi_{\nu,n+}$ is the optical path for the side of layer farthest from the detector.

With monchromatic transmittance defined as

$$\tau_\nu(\chi'_\nu, \chi_\nu) \equiv e^{-(\chi_\nu - \chi'_\nu)} = e^{\chi'_\nu - \chi_\nu}$$

We demonstrate the important principle about the product of transmittances:

$$\tau_\nu(a,b) = e^{a-b} \qquad \tau_\nu(b,c) = e^{b-c}$$

$$\tau_\nu(a,b)\tau_\nu(b,c) = e^{a-b}e^{b-c} = e^{a-c} = \tau_\nu(a,c)$$

Compare with mass path:

$$u(a,b) + u(b,c) = u(a,c)$$

Use the transmittance property:

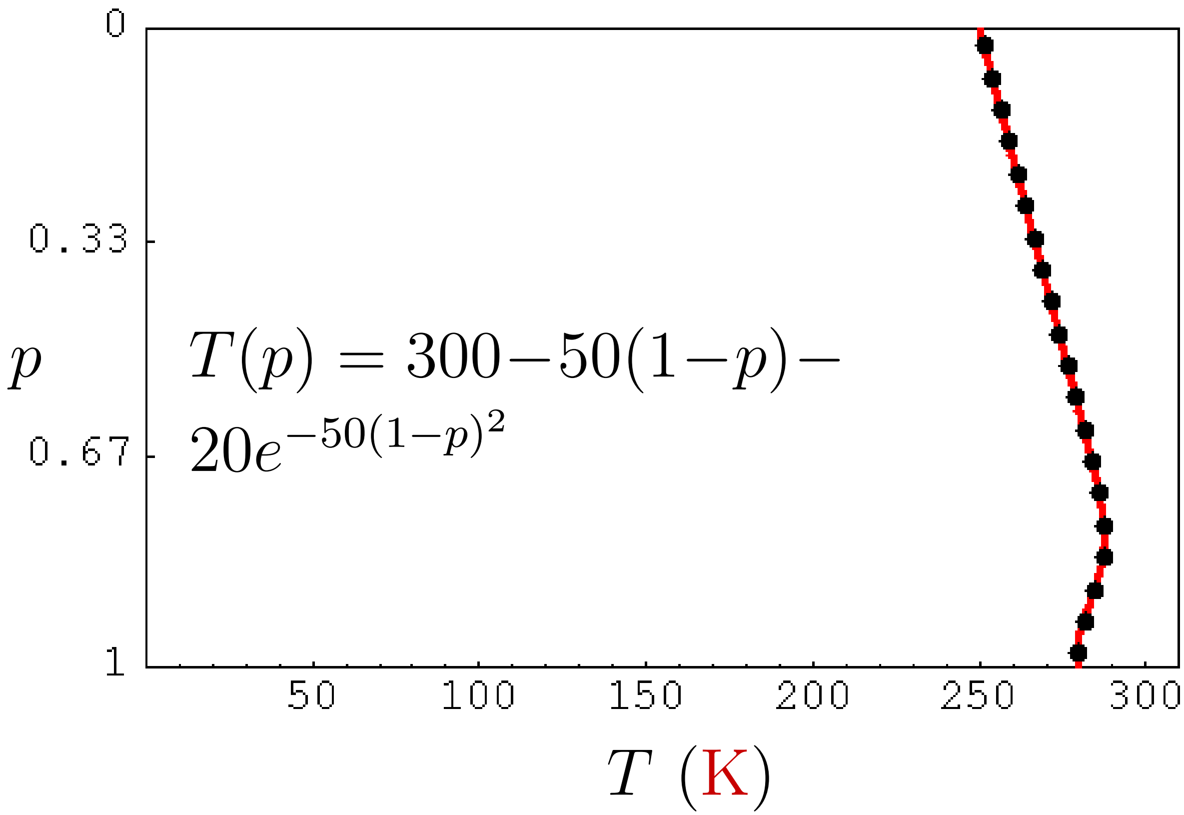
$$\tau_\nu(\chi_{\nu,n-}, \chi_\nu) = \tau_\nu(\chi_{\nu,n-}, \chi_{\nu,n+})\tau_\nu(\chi_{\nu,n+}, \chi_\nu)$$

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) +$$
$$\sum_{n=1}^{N} B_\nu(T_n)\left[\tau_\nu(\chi_{\nu,n+}, \chi_\nu) - \tau_\nu(\chi_{\nu,n-}, \chi_{\nu,n+})\tau_\nu(\chi_{\nu,n+}, \chi_\nu)\right]$$

$$L_\nu(\chi_\nu) = L_\nu(0)\tau_\nu(0, \chi_\nu) +$$
$$\sum_{n=1}^{N} B_\nu(T_n)\left[1 - \tau_\nu(\chi_{\nu,n-}, \chi_{\nu,n+})\right]\tau_\nu(\chi_{\nu,n+}, \chi_\nu)$$

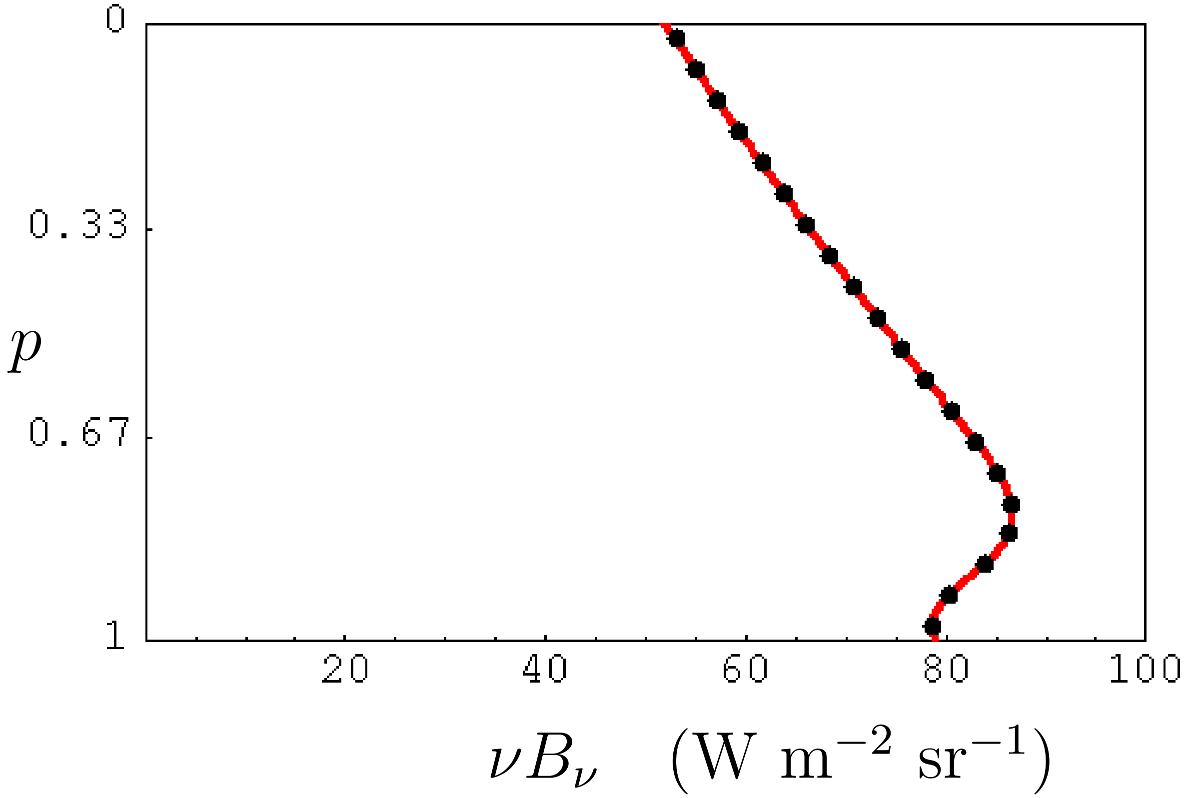
The term in [ ] is the *emittance* of the n'th layer.

Consider a very simple profile for $T(p)$ with a nocturnal boundary layer:



Will model with N=20 discrete isothermal layers.

We take $\nu = 2 \times 10^{13}$ Hz solely for the purpose of providing typical values for $B_\nu(T)$. But otherwise ν is variable:
we allow ν to vary so that we can study $\chi_{\nu\mathrm{m}}$ large or small.



The optical path through one of the N layers is:

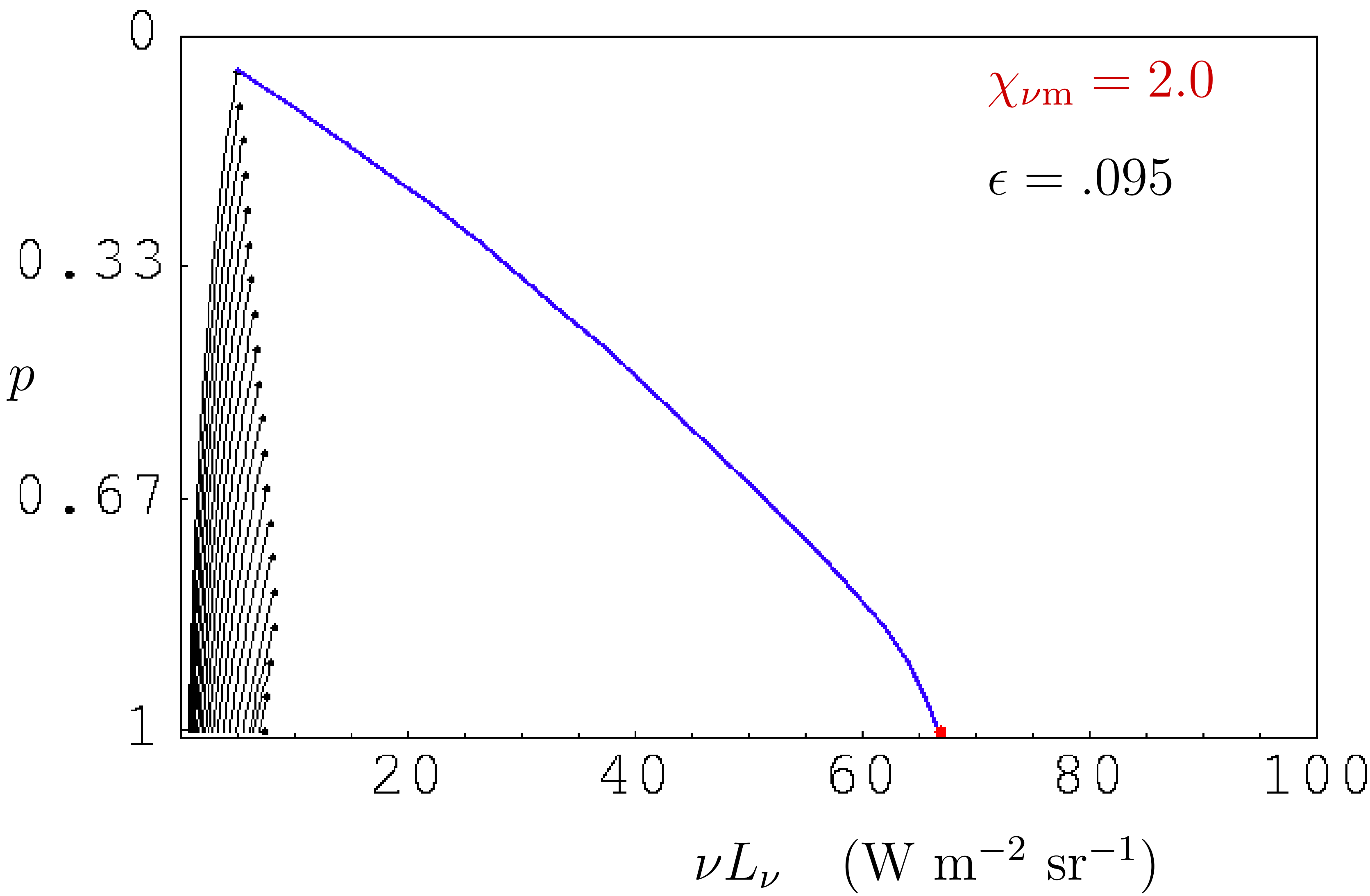
$$\Delta\chi_\nu = \frac{\chi_{\nu\mathrm{m}}}{N}$$

The spectral radiance emerging from the n'th layer is:

$$\nu L_\nu = \nu B_\nu(T_n)(1 - e^{-\Delta\chi_\nu}) \equiv \nu B_\nu(T_n)\epsilon$$

This spectral radiance will exponentially decay as it travels toward the ground.

Black: νL_ν emerging from the isothermal layers
Blue: total νL_ν. Red: total νL_ν at ground.



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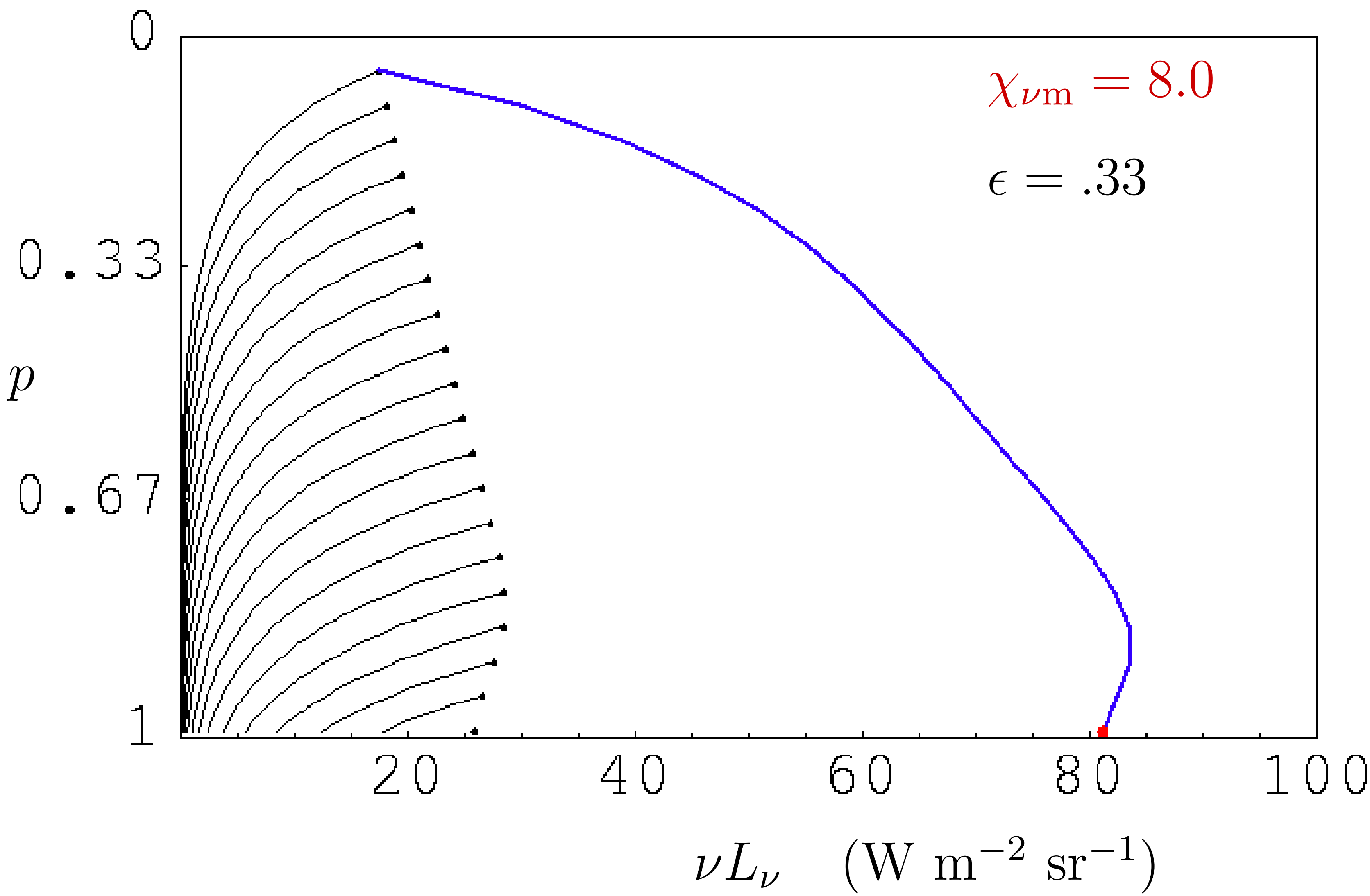

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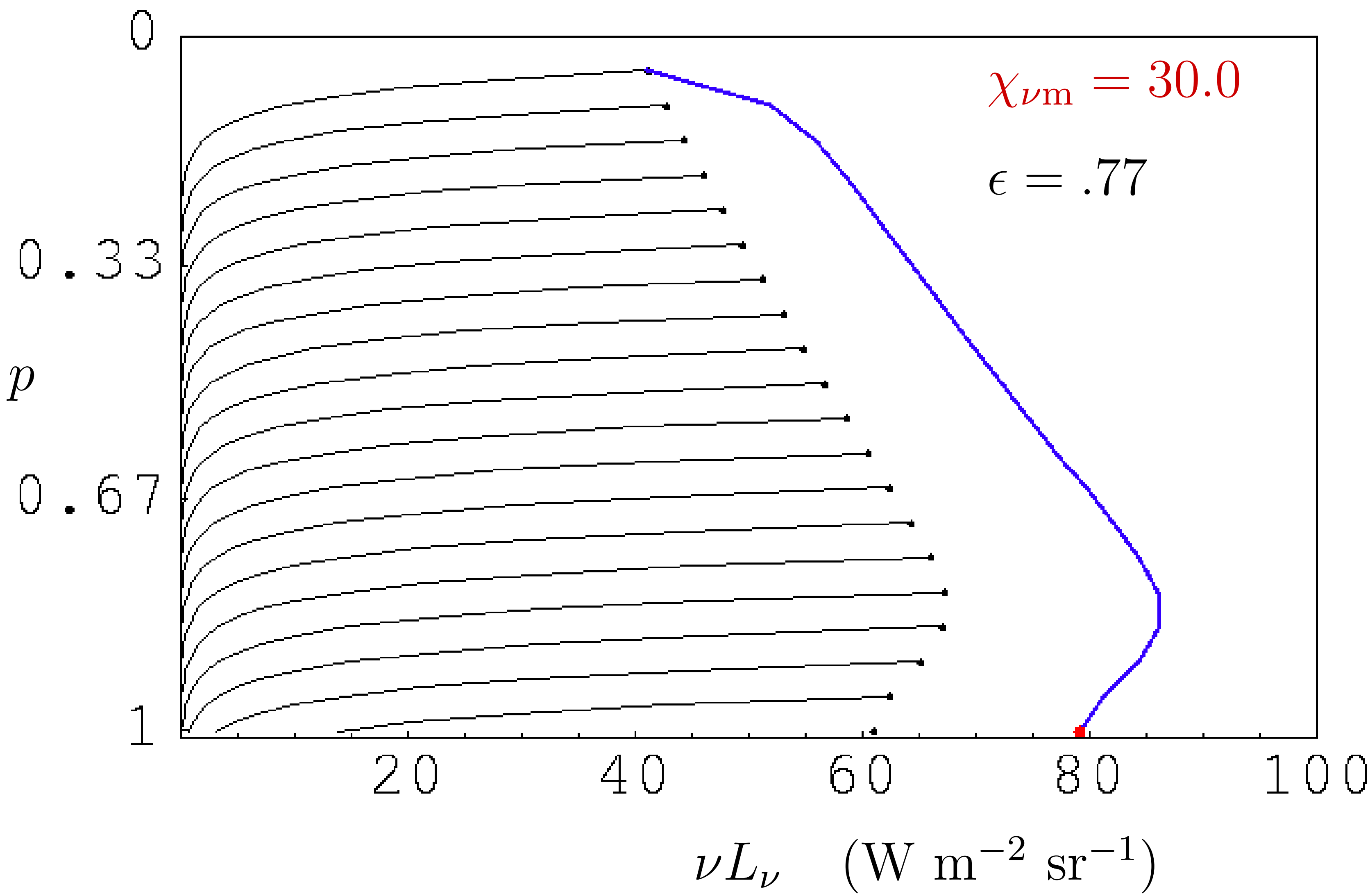

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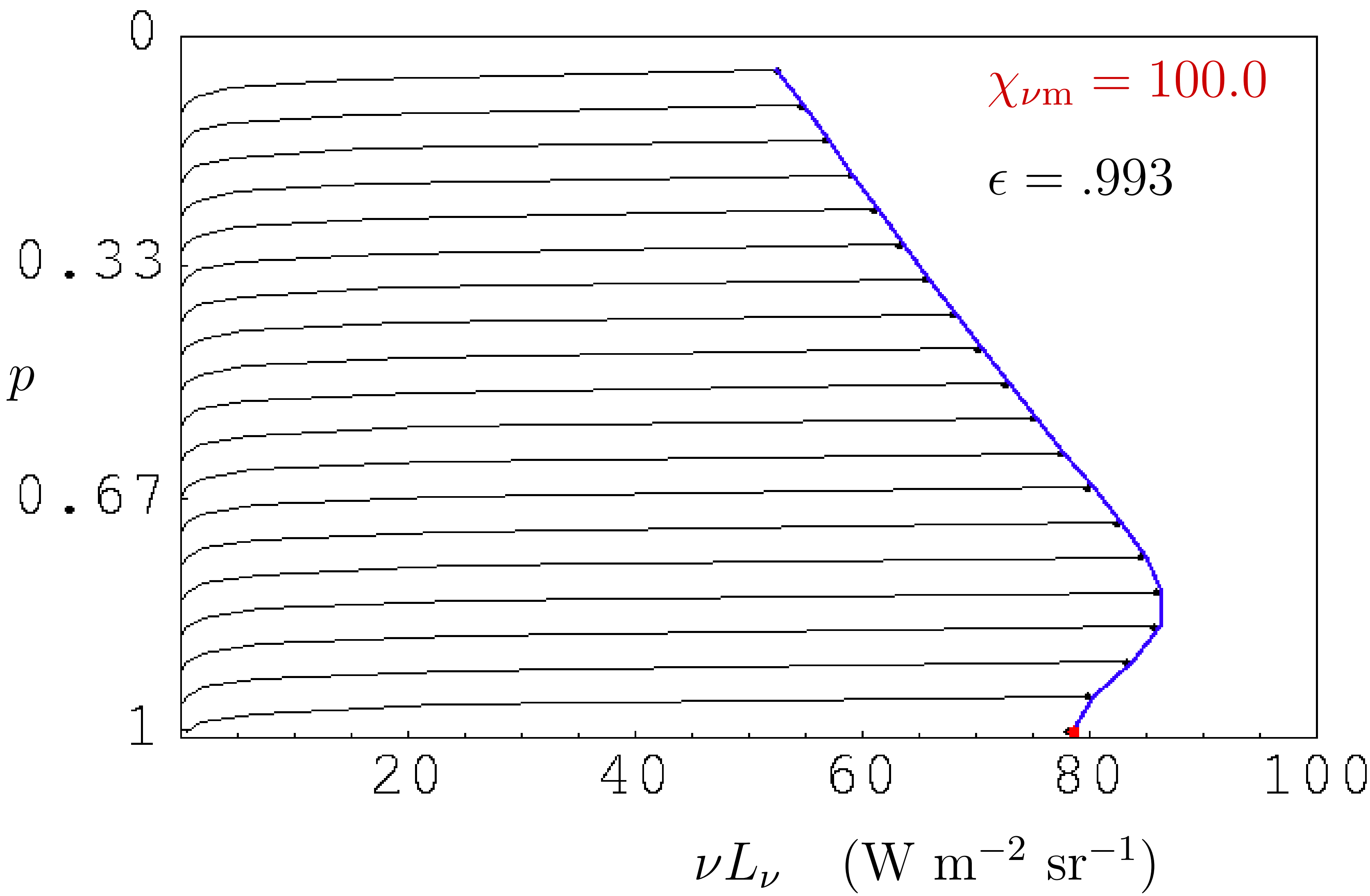

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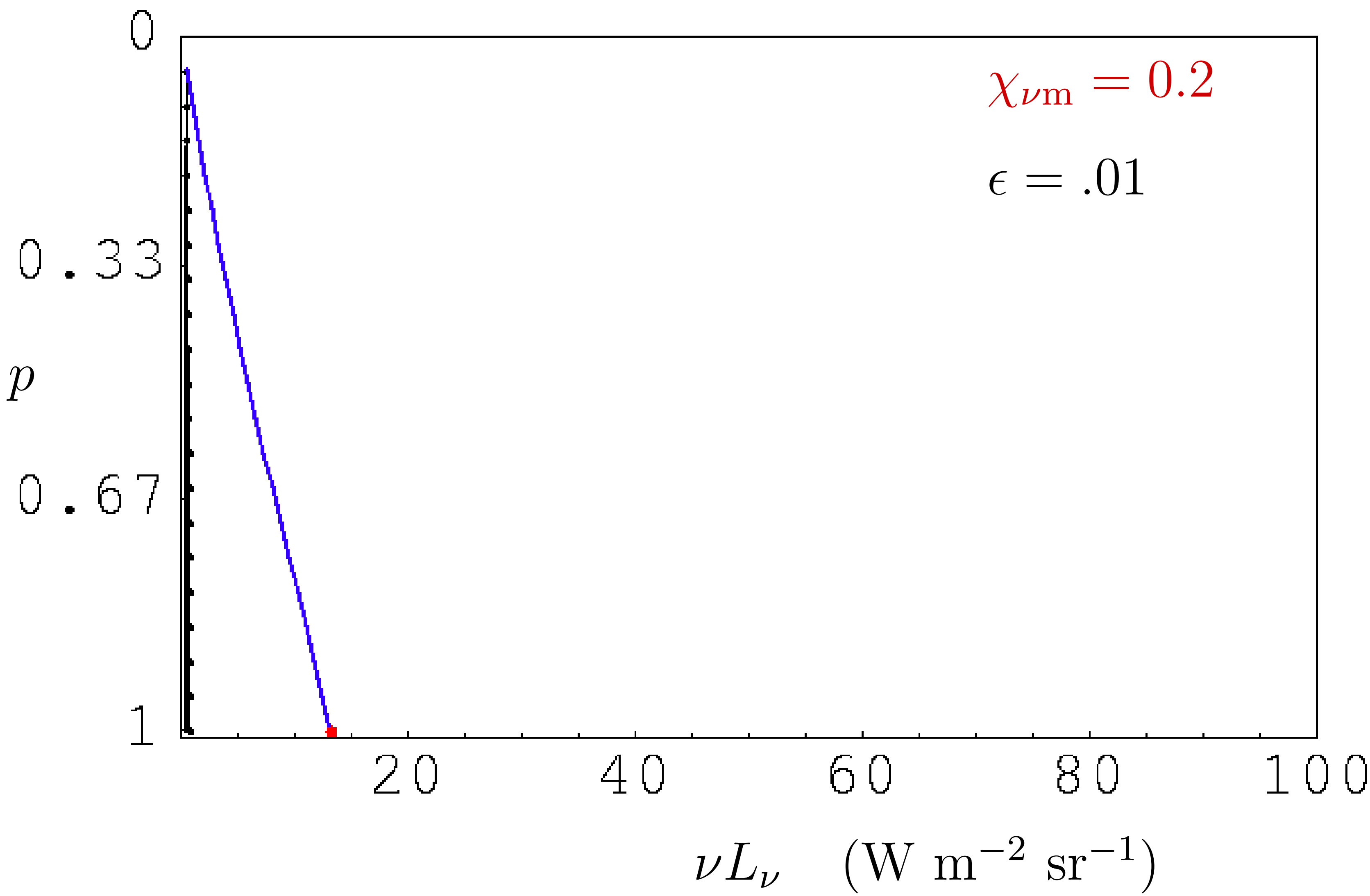

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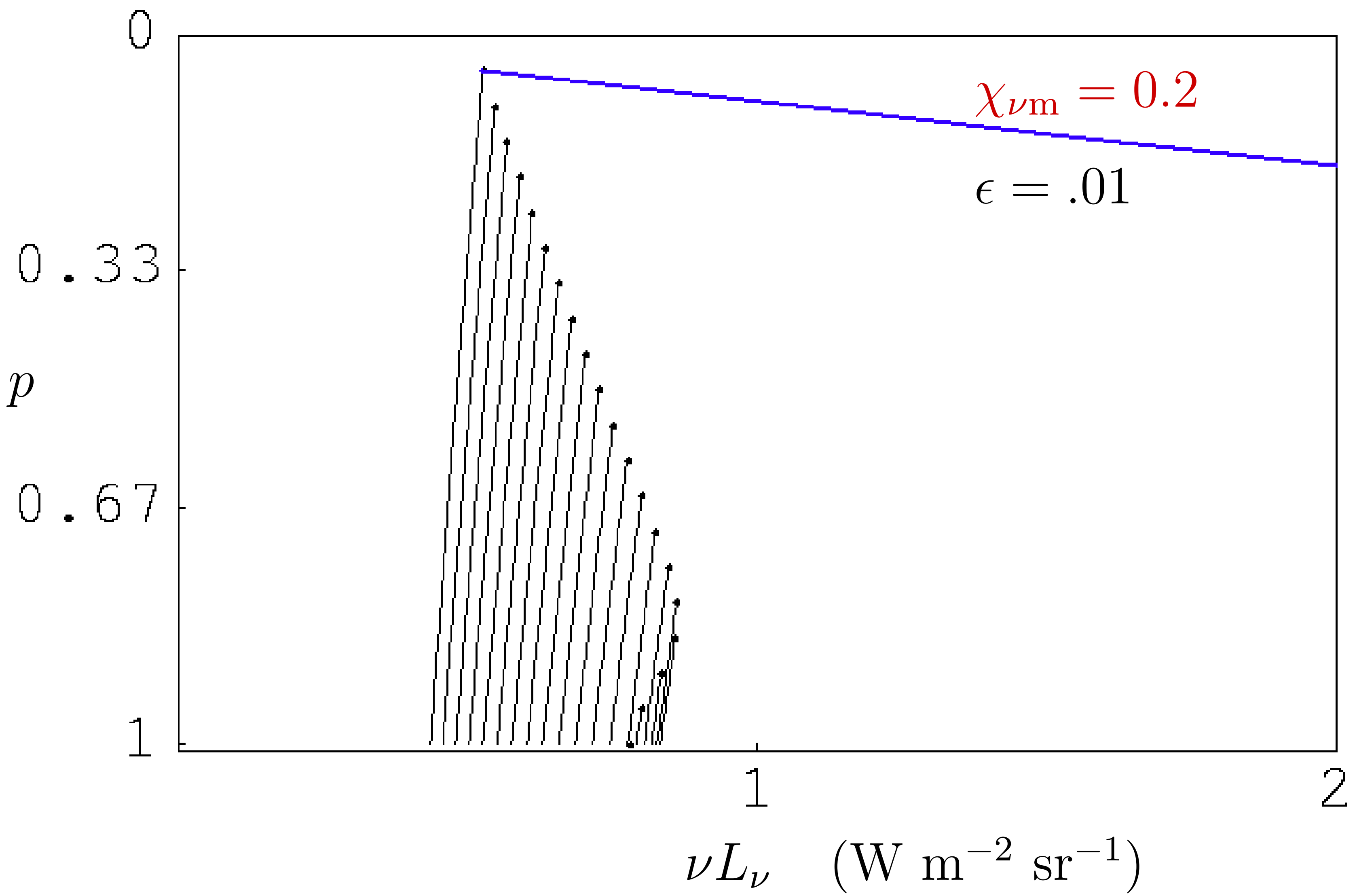

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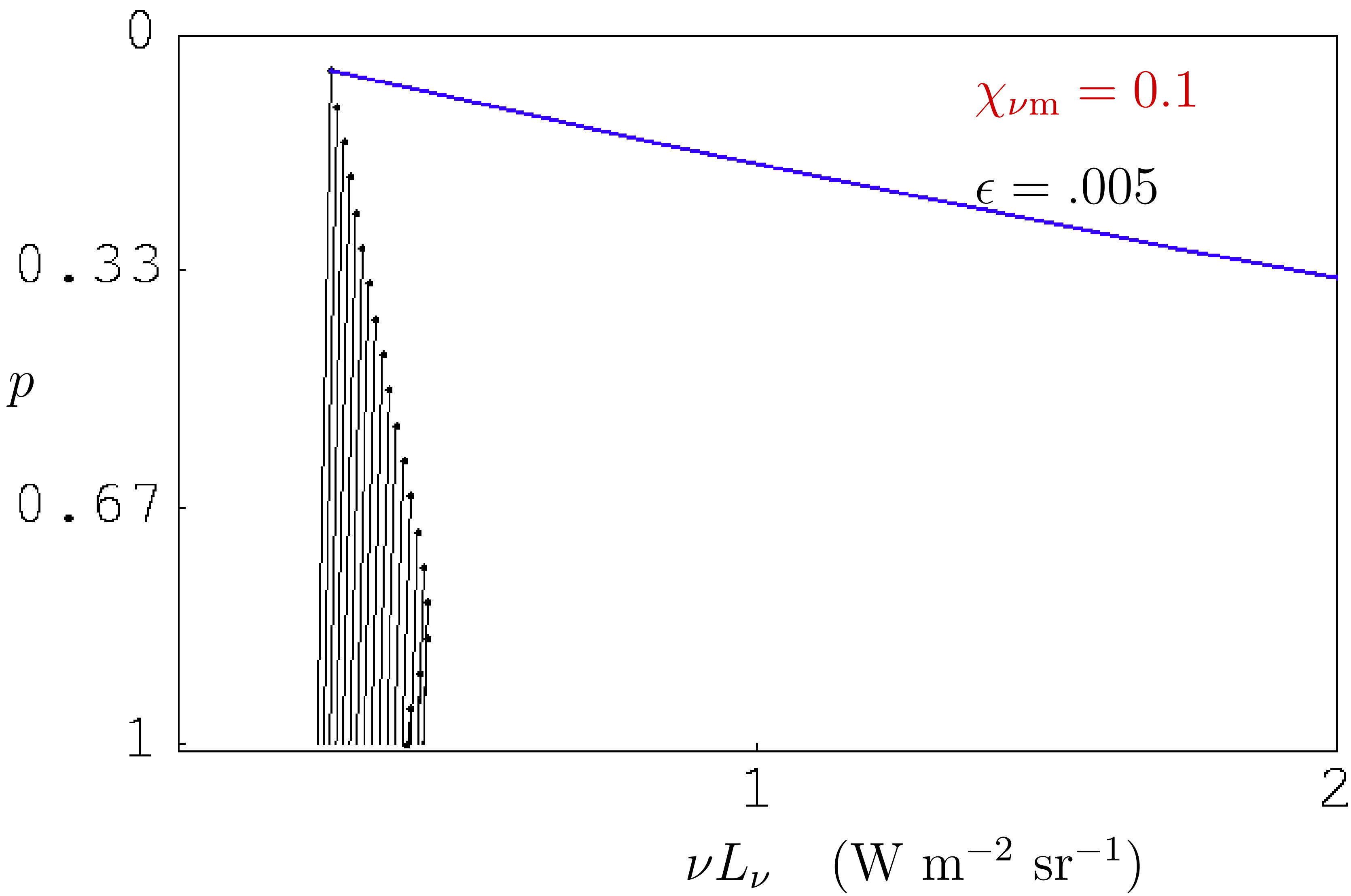
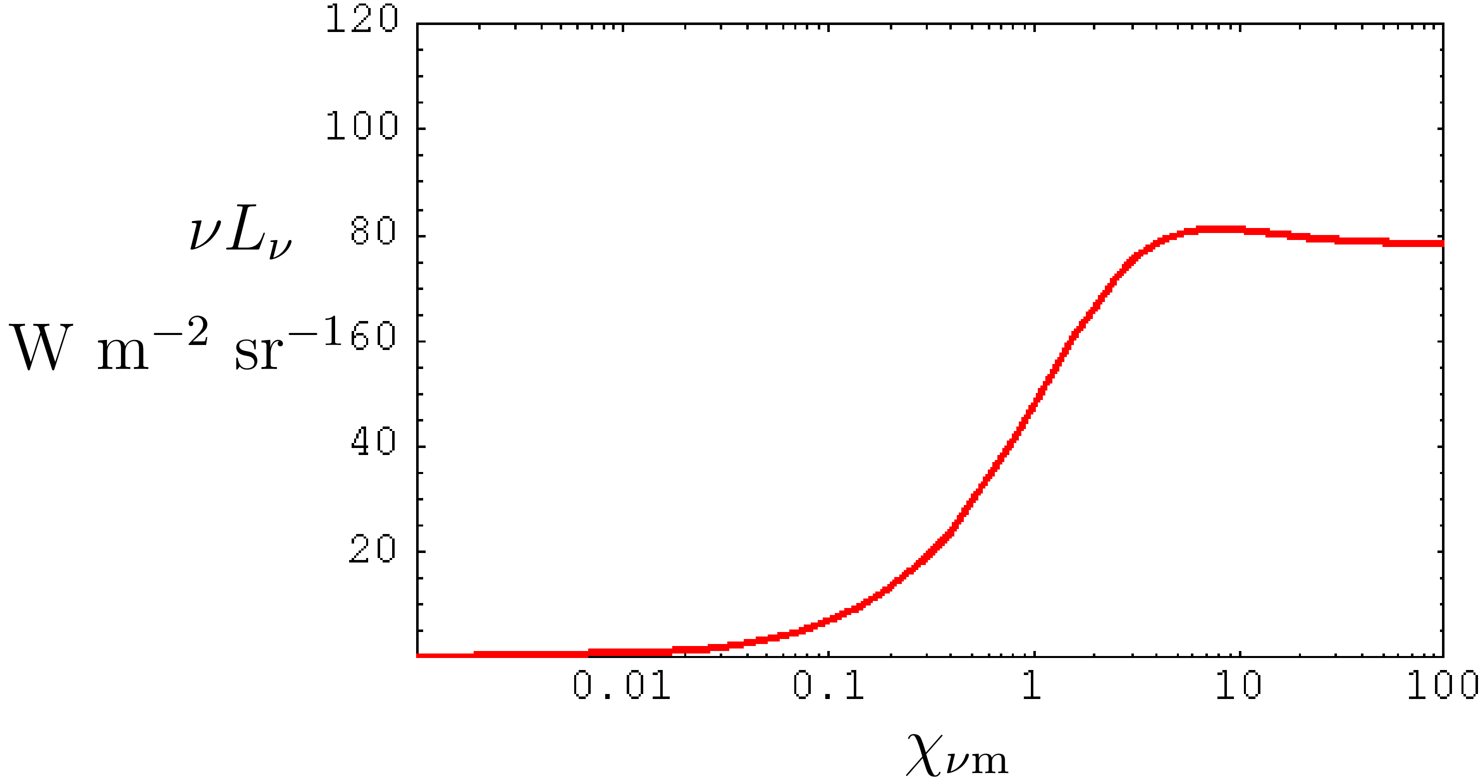
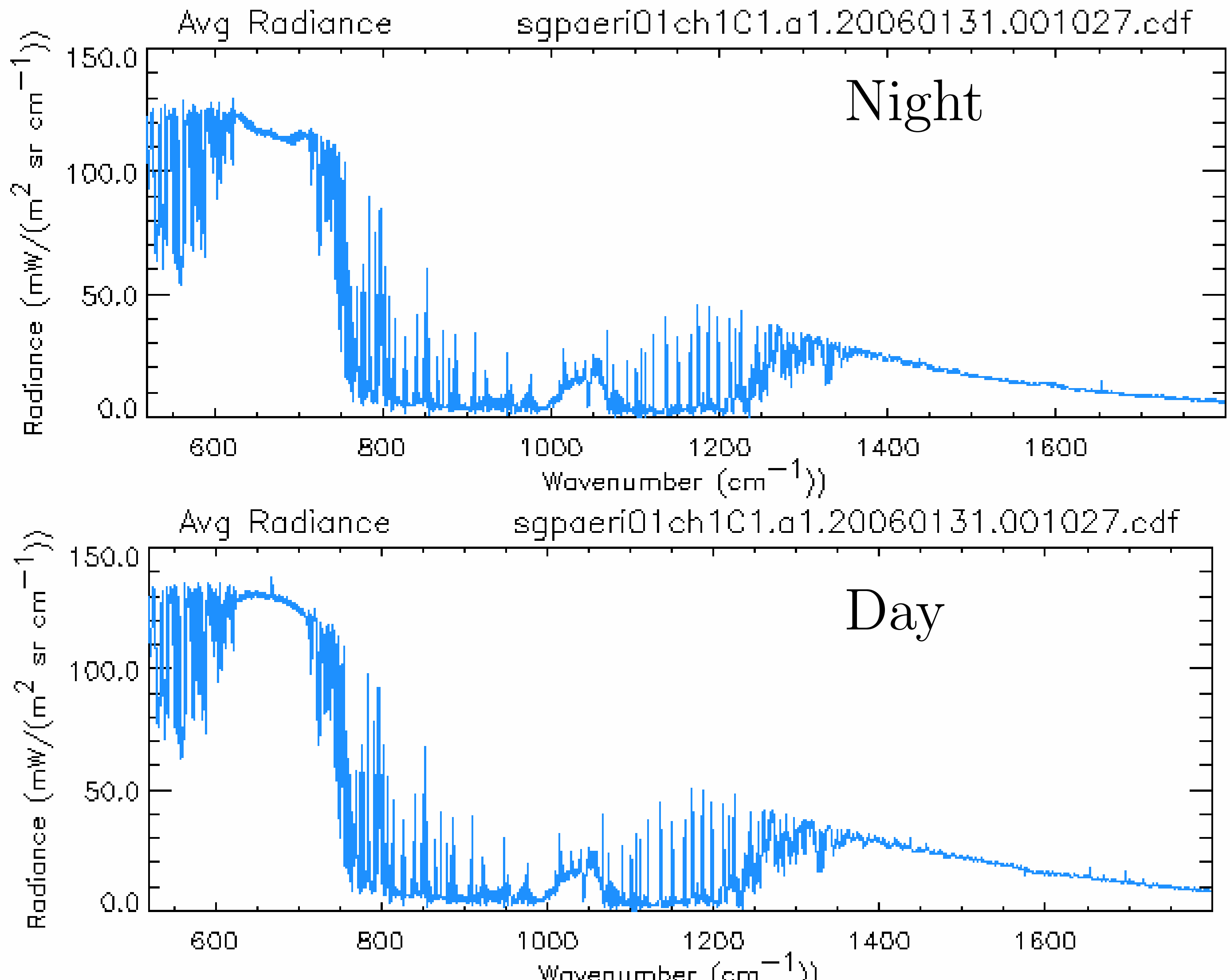


$\nu L_\nu(\chi_{\nu\mathrm{m}})$ into the ground from the discrete 20-layer model:



This plot is nearly indistinguishable from that of the complete model.



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