METR 5223: Atmospheric Radiation

Spherical Trigonometry and Solar Radiation on a Sphere

Lecture for Spring, 2009

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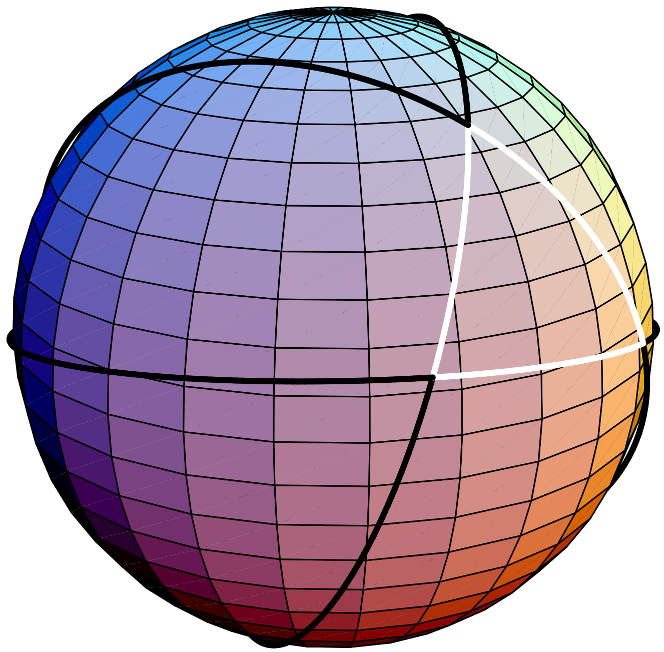
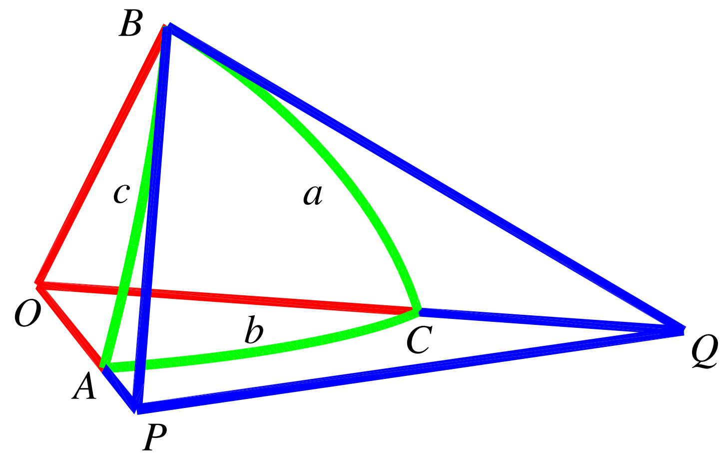
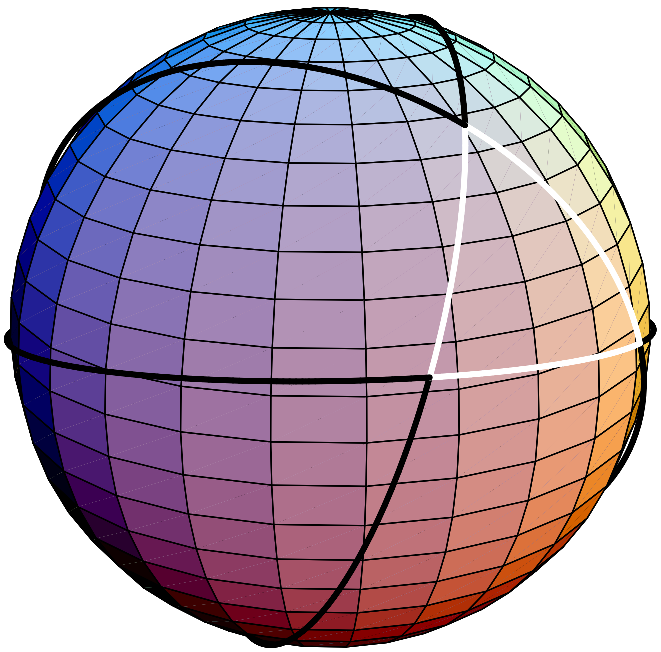
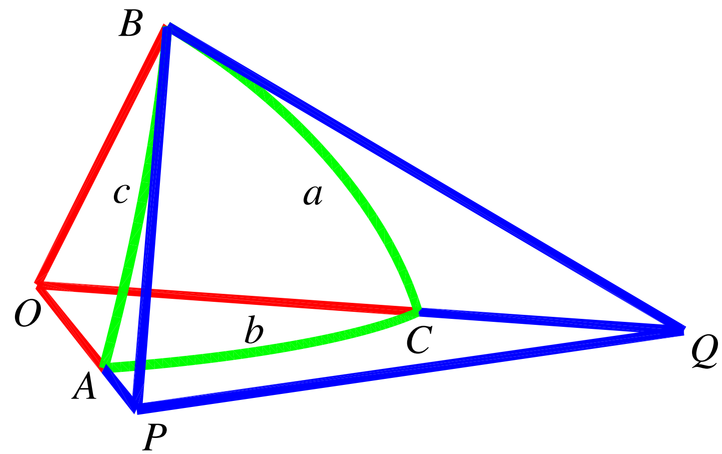


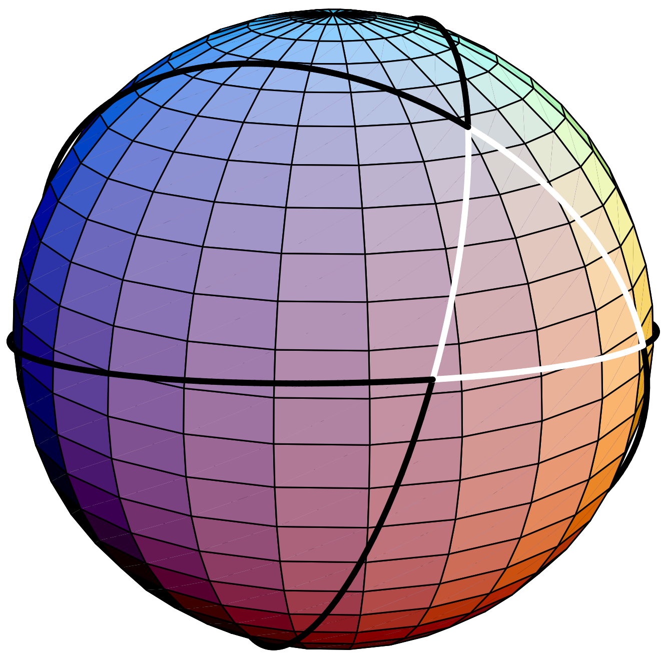


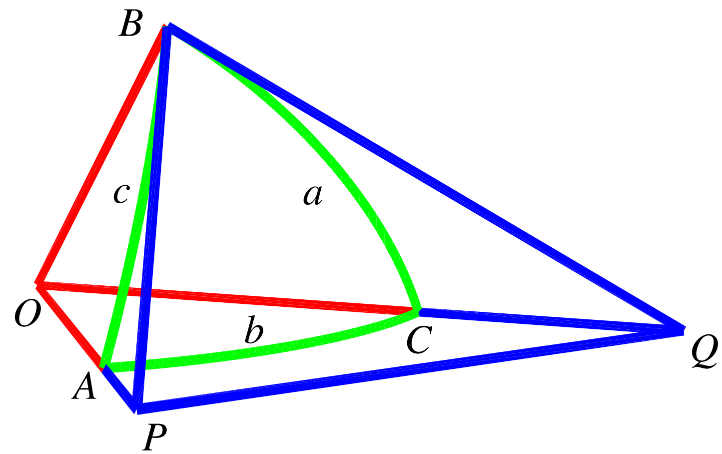
A great circle is formed by the intersection of a plane through the center of a sphere and the surface of the sphere. Three arcs of great circles can form a spherical triangle.



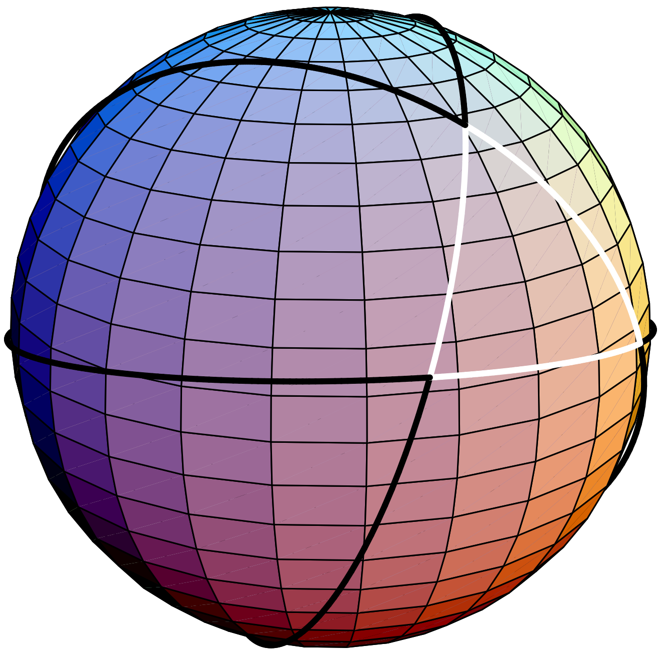


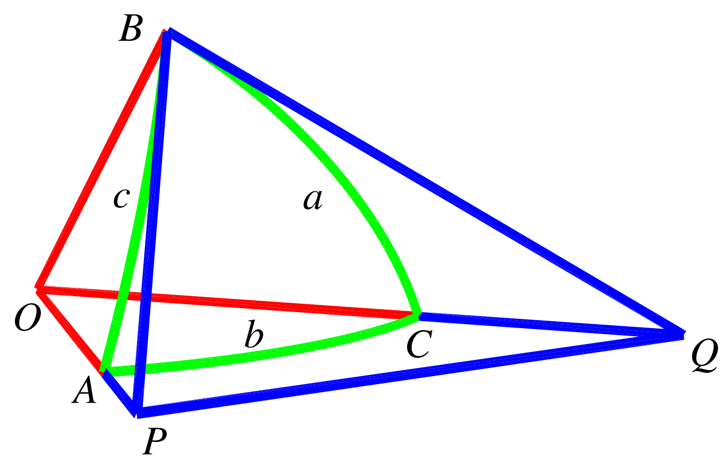
The points at the vertices of the spherical triangle are labeled A, B and C. The angles at those points have measure A, B and C. The sides opposite A, B and C have arc angle a, b and c.





The origin, or center of the sphere, is labeled O. Point P is at the intersection of a line through O and A and a plane tangent to the sphere at B. A similar definition applies to Q.



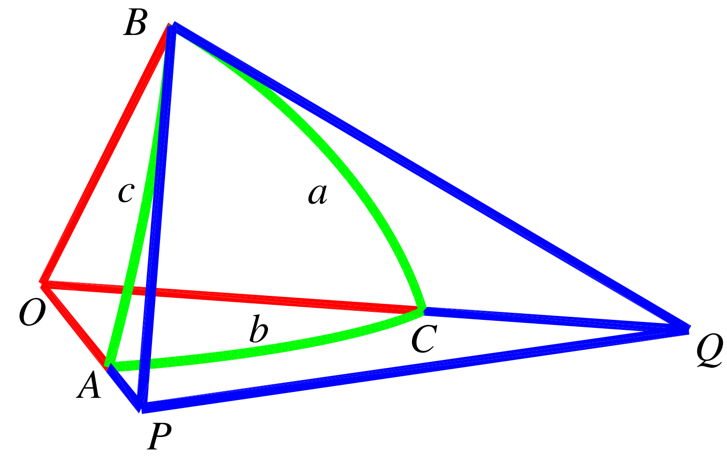


A is measured in a plane tangent to the sphere. a is the arc angle measured in a plane through the center of the sphere. A is related to a by the *law of cosines* for a spherical triangle:

$$\cos a = \cos c \ \cos b \ + \ \sin c \ \sin b \ \cos A.$$

First Proof

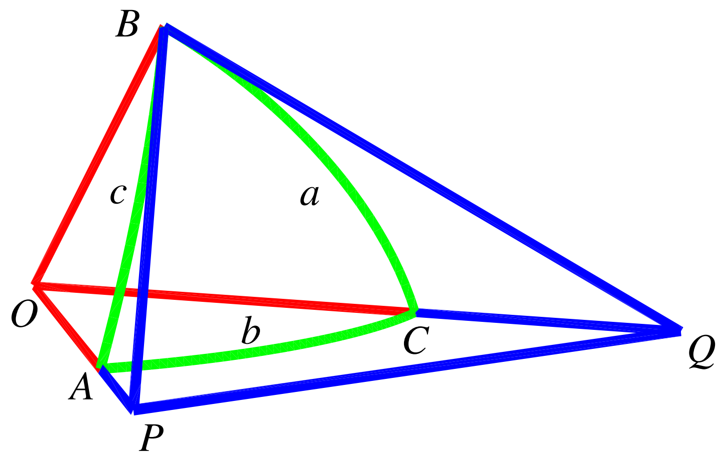
Angle A is the angle of intersection between planes OAB and OAC. This angle is the same as the angle between the normal vectors to those two planes.



$$\frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} \equiv \widehat{A} \quad \frac{\overrightarrow{OB}}{|\overrightarrow{OB}|} \equiv \widehat{B} \quad \frac{\overrightarrow{OC}}{|\overrightarrow{OA}|} \equiv \widehat{C}$$

The magnitude of $\widehat{A} \times \widehat{B}$ is $\sin c$ and the magnitude of $\widehat{A} \times \widehat{C}$ is $\sin b$.

$$\frac{\widehat{A} \times \widehat{B}}{\sin c} \cdot \frac{\widehat{A} \times \widehat{C}}{\sin b} = \cos A$$



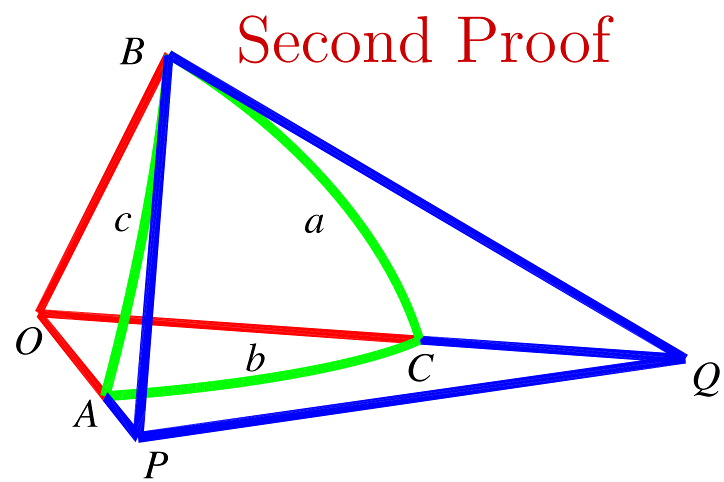
$$(\widehat{A} \times \widehat{B}) \cdot (\widehat{A} \times \widehat{C}) = \widehat{B} \cdot \widehat{C}\ \widehat{A} \cdot \widehat{A} - \widehat{B} \cdot \widehat{A}\ \widehat{A} \cdot \widehat{C}$$

$$(\widehat{A} \times \widehat{B}) \cdot (\widehat{A} \times \widehat{C}) = \cos a - \cos c\ \cos b$$

$$\cos a - \cos c\ \cos b = \cos A\ \sin c\ \sin b$$

or

$$\cos a = \cos c\ \cos b\ +\ \sin c\ \sin b\ \cos A.$$



Applying the law of cosines for plane triangles OPQ and BPQ we have:

$$(PQ)^2 = (OQ)^2 + (OP)^2 - 2(OQ)(OP)\cos b$$

and

$$(PQ)^2 = (BQ)^2 + (BP)^2 - 2(BQ)(BP)\cos B$$

Taking the difference of these equations gives

$$\begin{aligned} 0 &= (OQ)^2 - (BQ)^2 + (OP)^2 - (BP)^2 \\ &\quad -2(OQ)(OP)\cos b + 2(BQ)(BP)\cos B \end{aligned}$$

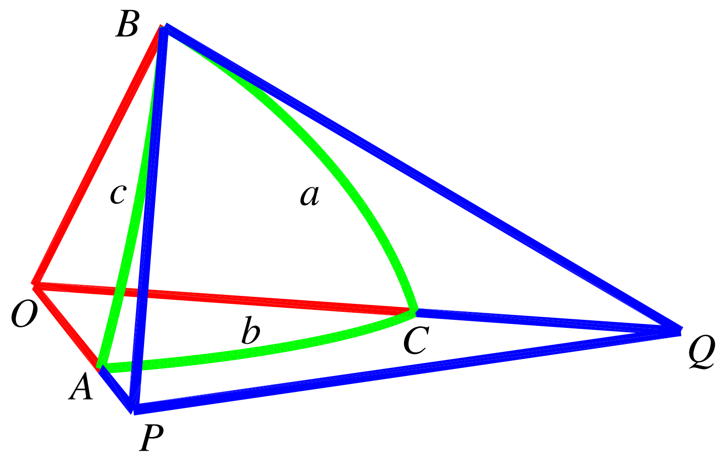
With the Pythagorean theorem giving

$$(OQ)^2 = (BQ)^2 + (OB)^2$$
$$(OP)^2 = (BP)^2 + (OB)^2$$

$$0 = 2(OB)^2 - 2(OQ)(OP)\cos b + 2(BQ)(BP)\cos B$$

and thus

$$\cos b = \frac{OB}{OQ} \cdot \frac{OB}{OP} + \frac{BQ}{OQ} \cdot \frac{BP}{OP} \cos B$$

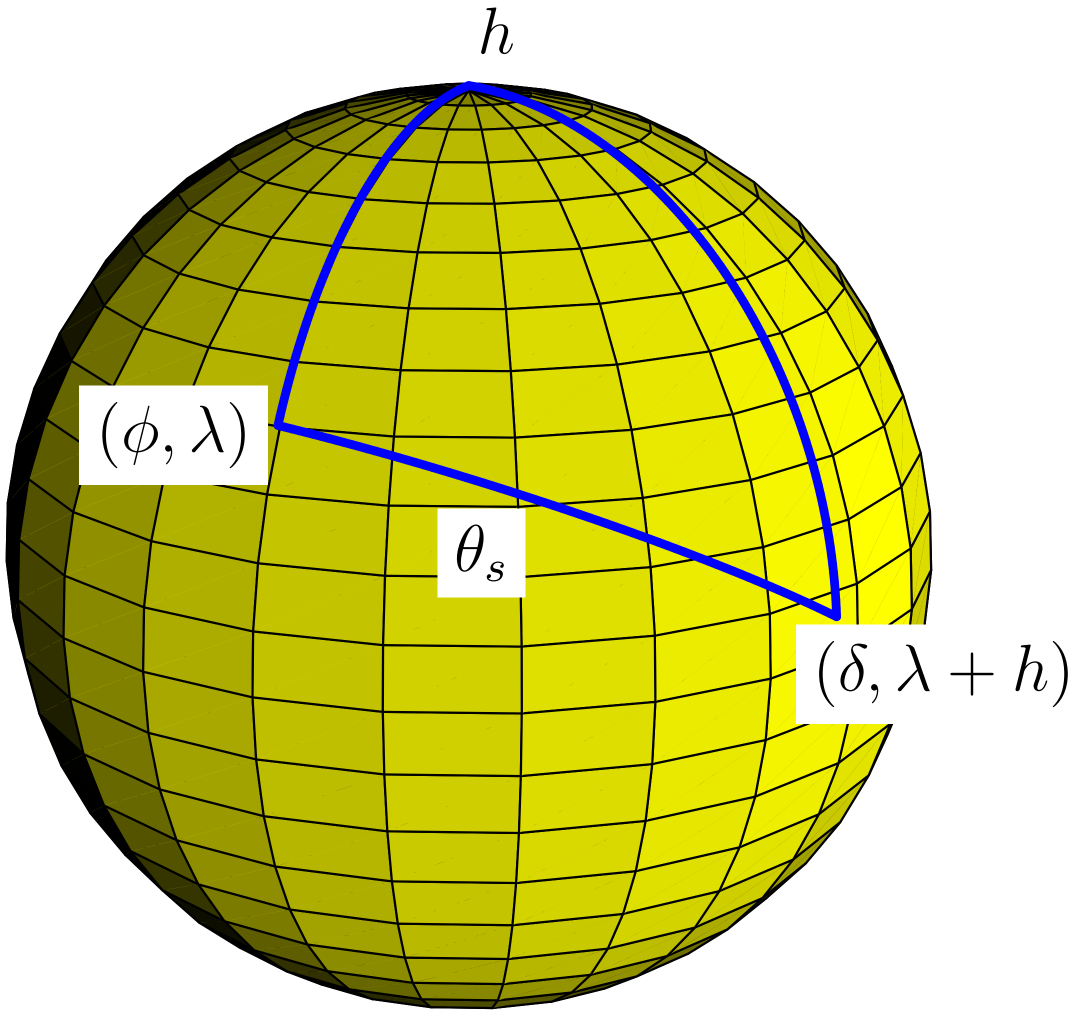


Noting the right triangles $\triangle OBP$ and $\triangle OBQ$,

$$\cos b = \frac{OB}{OQ} \cdot \frac{OB}{OP} + \frac{BQ}{OQ} \cdot \frac{BP}{OP} \cos B$$

is written as:

$$\cos b = \cos c \;\cos a \;+\; \sin c \;\sin a \;\cos B.$$



Calculate the zenith angle θ_s at a lat-lon point (ϕ, λ). The subsolar point is at latitude δ (the *solar declination angle*) and longitude $\lambda + h$ where h is the *hour angle*.

$$\cos a = \cos c \ \cos b \ + \ \sin c \ \sin b \ \cos A.$$

$$\cos \theta_s = \cos\left(\frac{\pi}{2} - \phi\right) \ \cos\left(\frac{\pi}{2} - \delta\right) + \sin\left(\frac{\pi}{2} - \phi\right) \ \sin\left(\frac{\pi}{2} - \delta\right) \ \cos h$$

$$\cos\theta_s = \sin\phi\sin\delta + \cos\phi\cos\delta\cos h$$

The distance of Earth from the sun is R_E. The mean distance is R_o. The solar flux density into a plane tangent to the sphere, but at the top of the atmosphere:

$$Q = S_o \frac{R_o^2}{R_E^2}\cos\theta_s \text{ when } \cos\theta_s > 0$$

$$Q = 0 \text{ when } \cos\theta_s \le 0$$

Sunrise occurs at hour angle h_o, (and sunset occurs at hour angle $-h_o$), when $\theta_s = \frac{\pi}{2}$:

$$\sin\phi\sin\delta + \cos\phi\cos\delta\cos h_o = 0$$

The average value of Q over a day is:

$$\overline{Q}^{day} = \frac{\int_{h=\pi}^{h=-\pi} Q dh}{\int_{h=\pi}^{h=-\pi} dh}$$

Now $\int_{h=\pi}^{h=-\pi} dh = -2\pi$ and, if $\frac{R_o^2}{R_E^2}$ is nearly constant over the course of a day,

$$\begin{aligned}
\int_{h=\pi}^{h=-\pi} Q dh &= \int_{h=h_o}^{h=-h_o} Q dh = S_o \frac{R_o^2}{R_E^2} \int_{h=h_o}^{h=-h_o} \cos\theta_s dh \\
&= S_o \frac{R_o^2}{R_E^2} \left[h \sin\phi \sin\delta + \cos\phi \cos\delta \sin h\right]_{h=h_o}^{h=-h_o} \\
&= -2S_o \frac{R_o^2}{R_E^2} \left[h_o \sin\phi \sin\delta + \cos\phi \cos\delta \sin h\right]
\end{aligned}$$

Summary:

$$\overline{Q}^{day} = \frac{S_o}{\pi}\frac{R_o^2}{R_E^2}\left(h_o \sin\phi \sin\delta + \cos\phi\cos\delta\sin h_o\right)$$

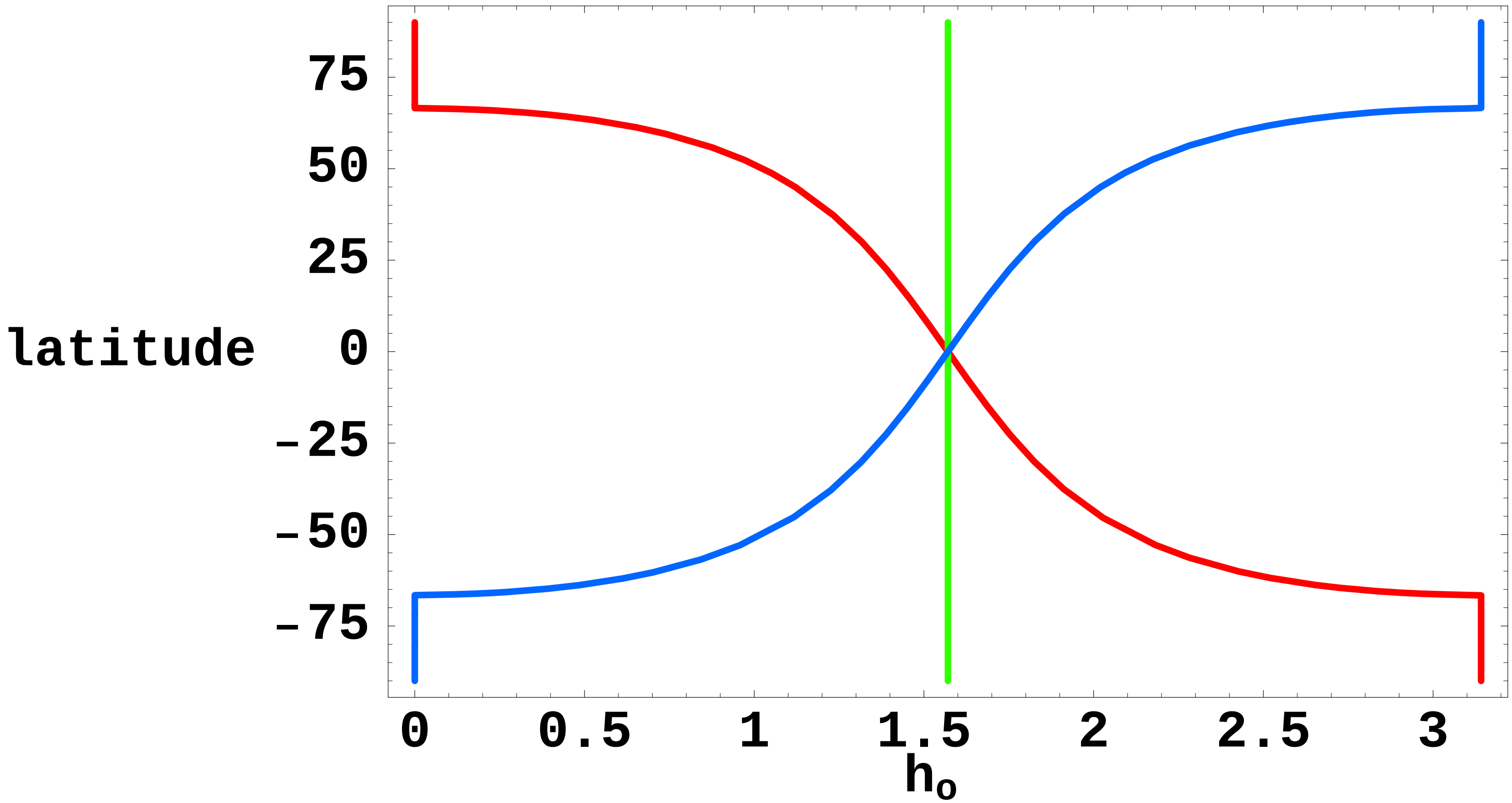
where

$$\sin\phi\sin\delta + \cos\phi\cos\delta\cos h_o = 0$$

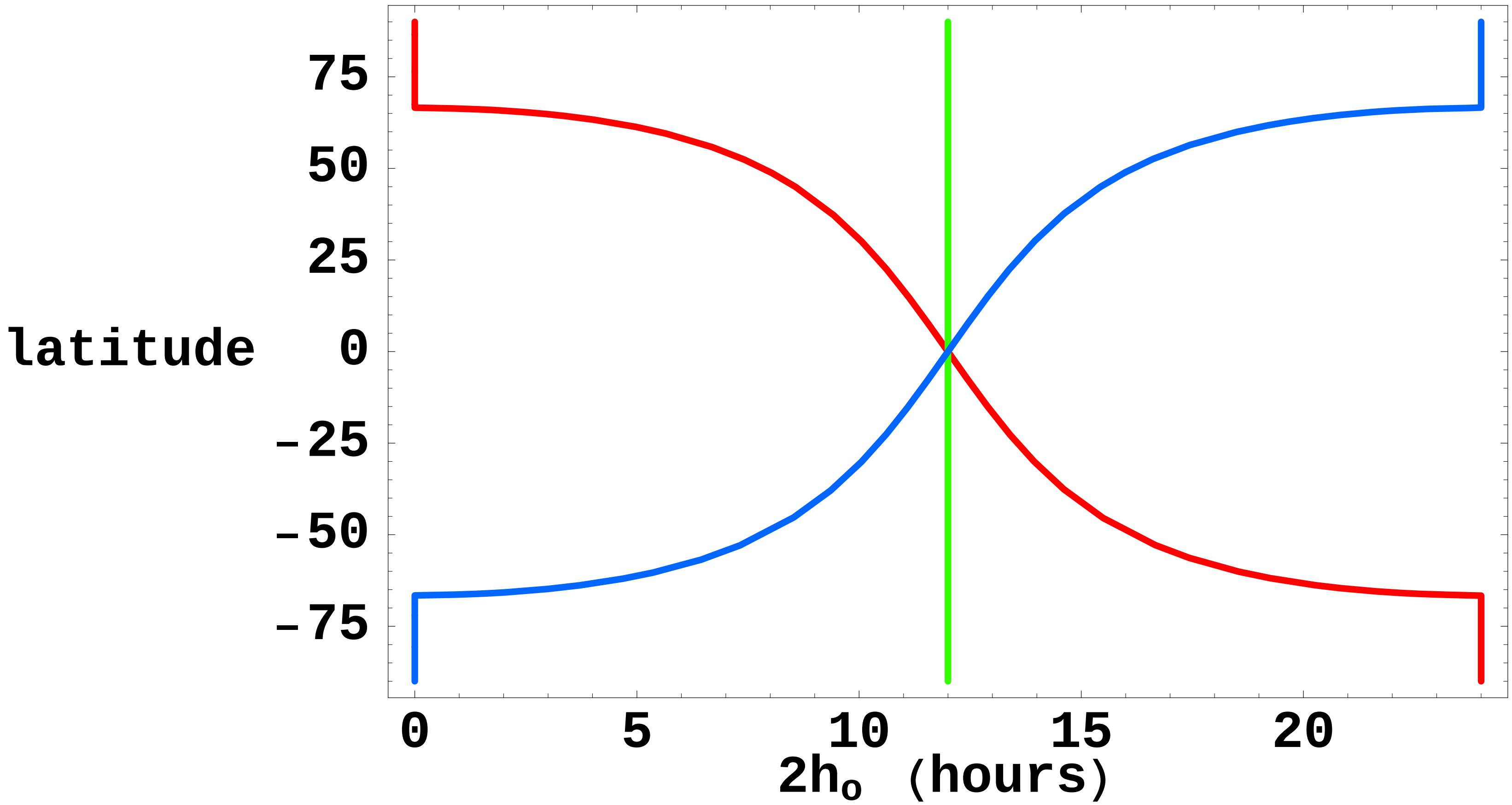
(But sometimes θ_s is never 0 during a day, and $h_o = \pi$ or $h_o = 0$.)

Keep it simple for a first calculation of $\overline{Q}^{day}$:

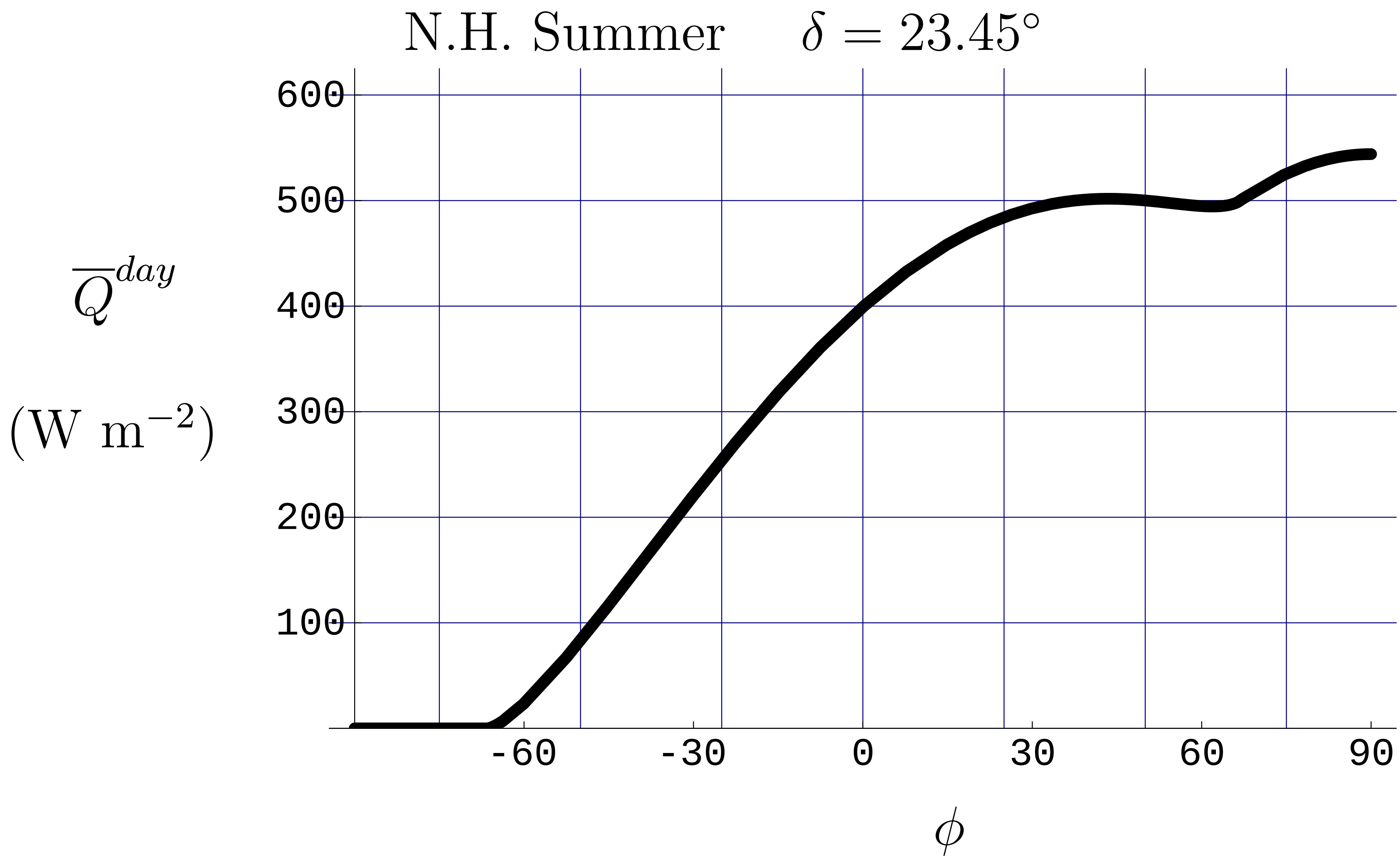
1. Assume $\frac{R_o^2}{R_E^2} = 1$ exactly.
2. Know the current obliquity of Earth is 23.45°. δ oscillates between $\pm 23.45°$.
3. Calculate $\overline{Q}^{day}$ at the solstices and equinoxes.



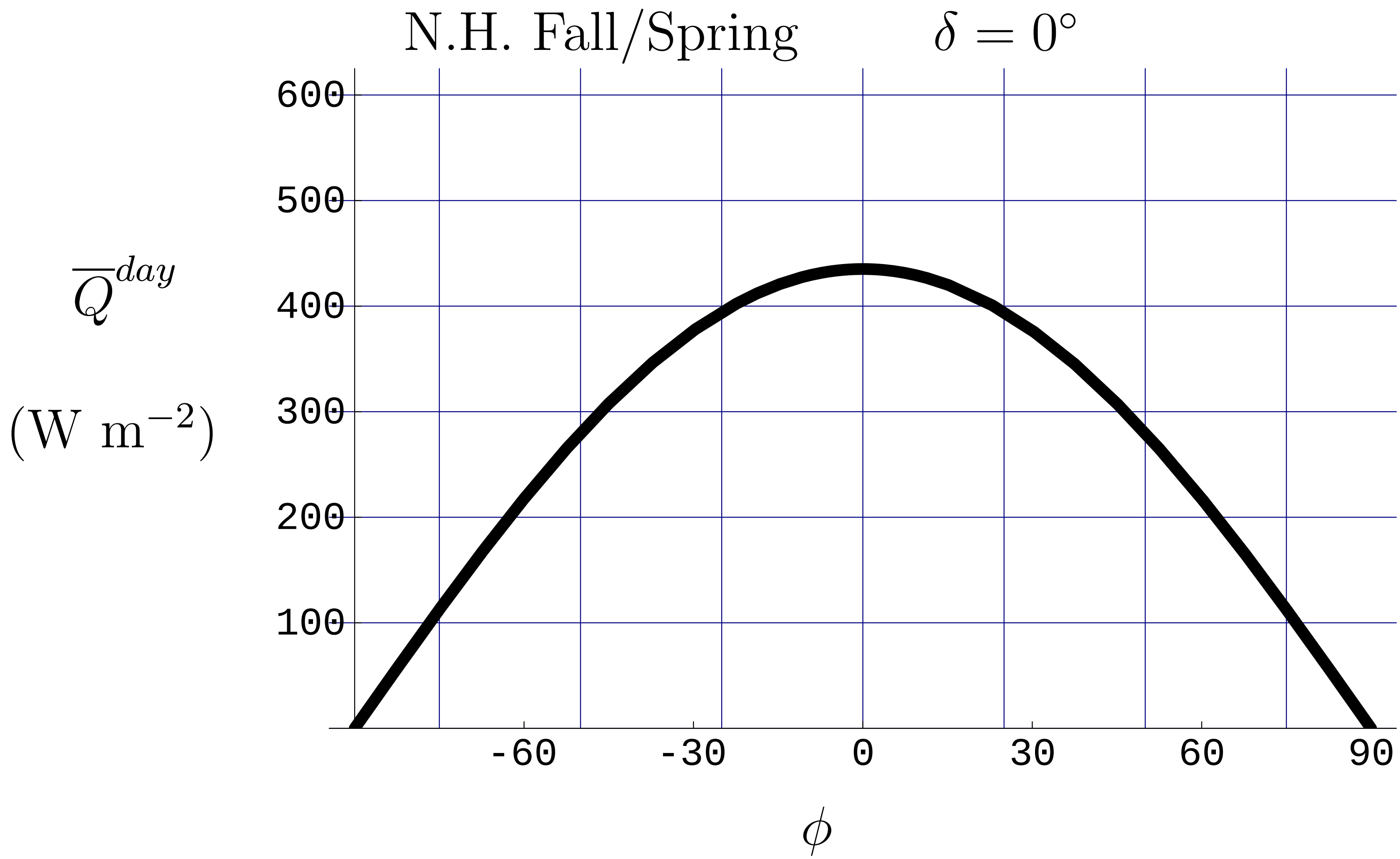
summer solstice, equinox, winter solstice



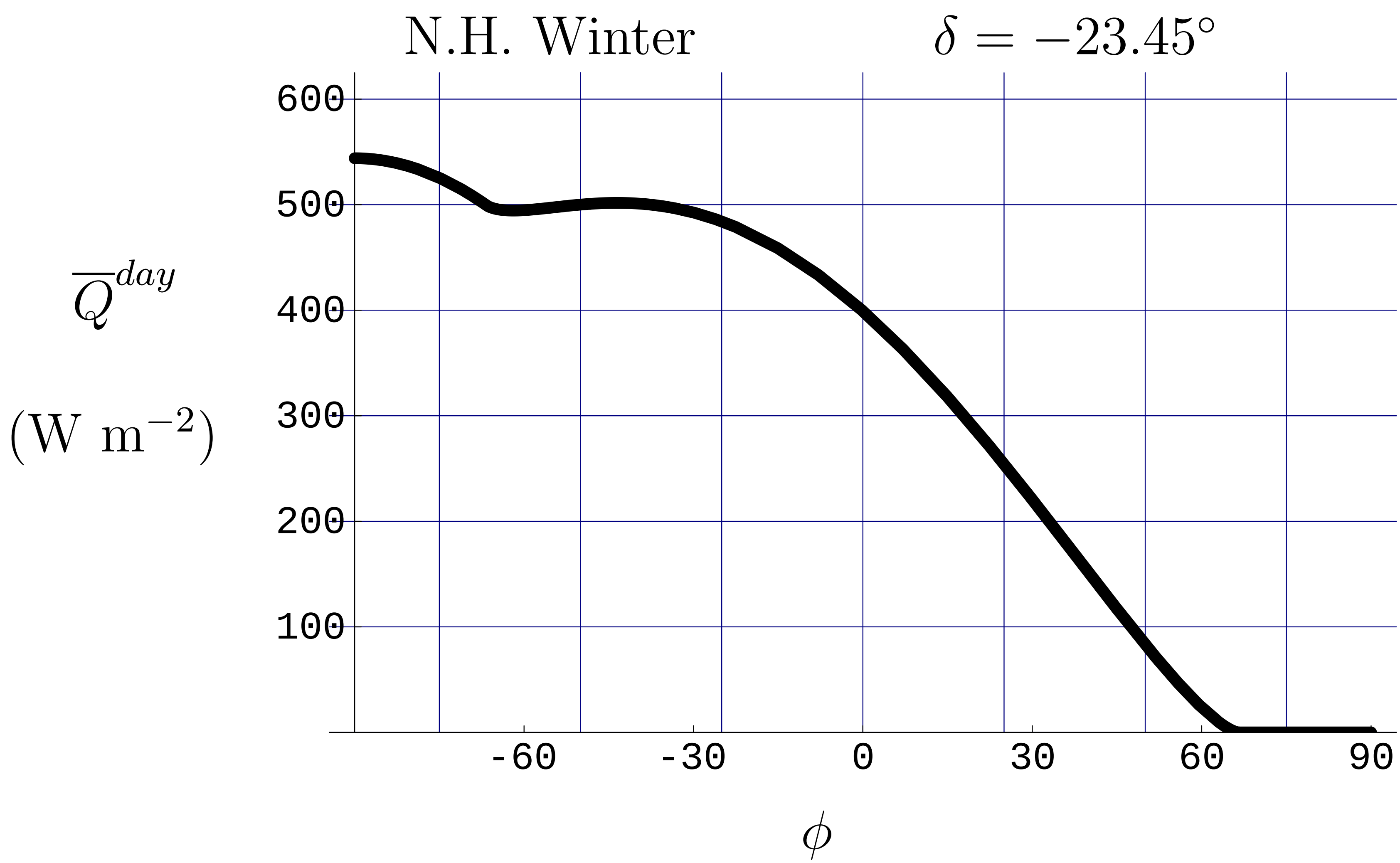
summer solstice, equinox, winter solstice



N.H. Summer $\delta = 23.45°$
$\overline{Q}^{day}$
(W m^{-2})
600
500
400
300
200
100
-60
-30
0
30
60
90
ϕ

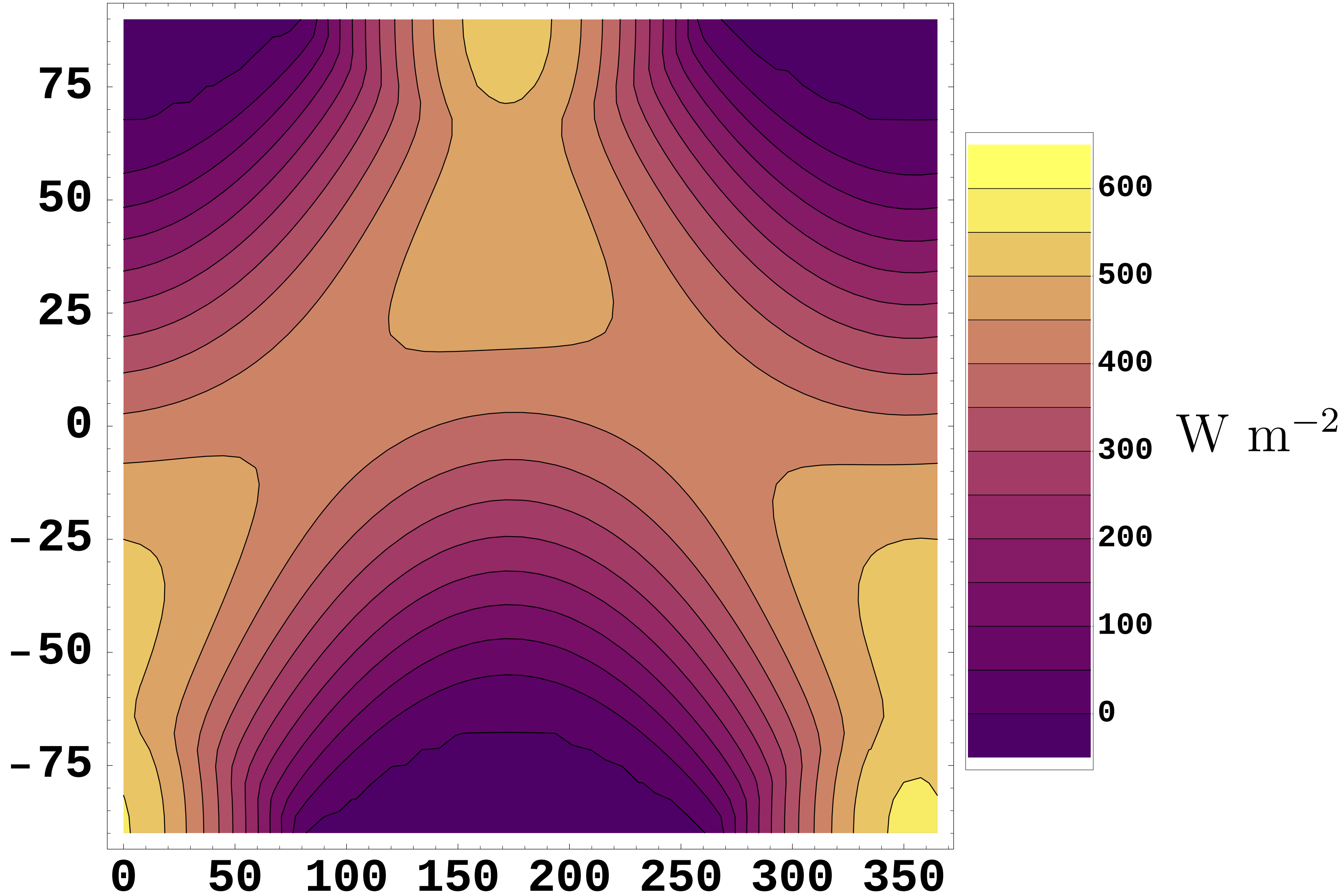


N.H. Fall/Spring $\delta = 0°$
$\overline{Q}^{day}$
(W m^{-2})
600
500
400
300
200
100
-60
-30
0
30
60
90
ϕ

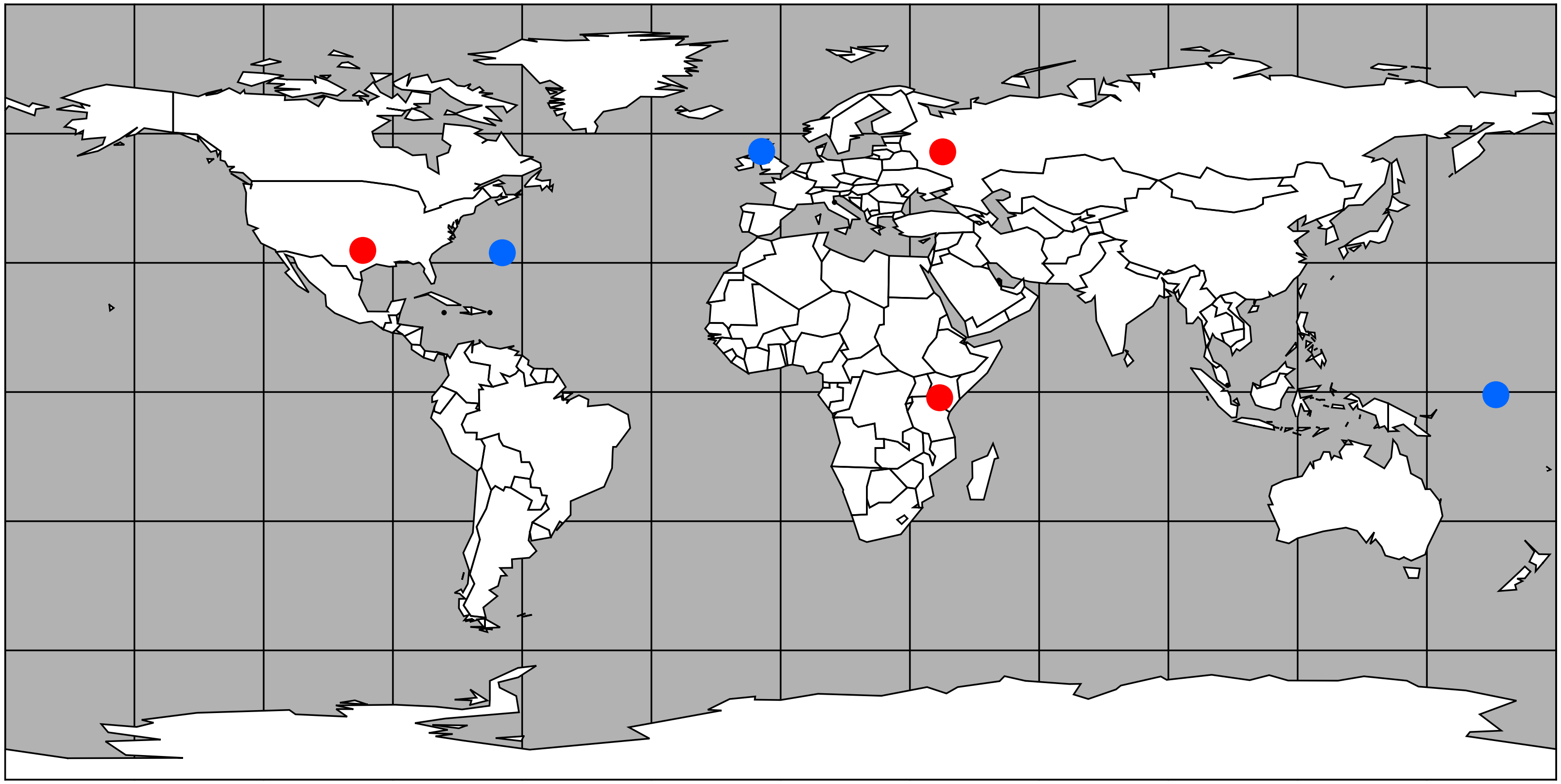


N.H. Winter
$\delta = -23.45°$
$\overline{Q}^{day}$
(W m^{-2})
600
500
400
300
200
100
-60
-30
0
30
60
90
ϕ

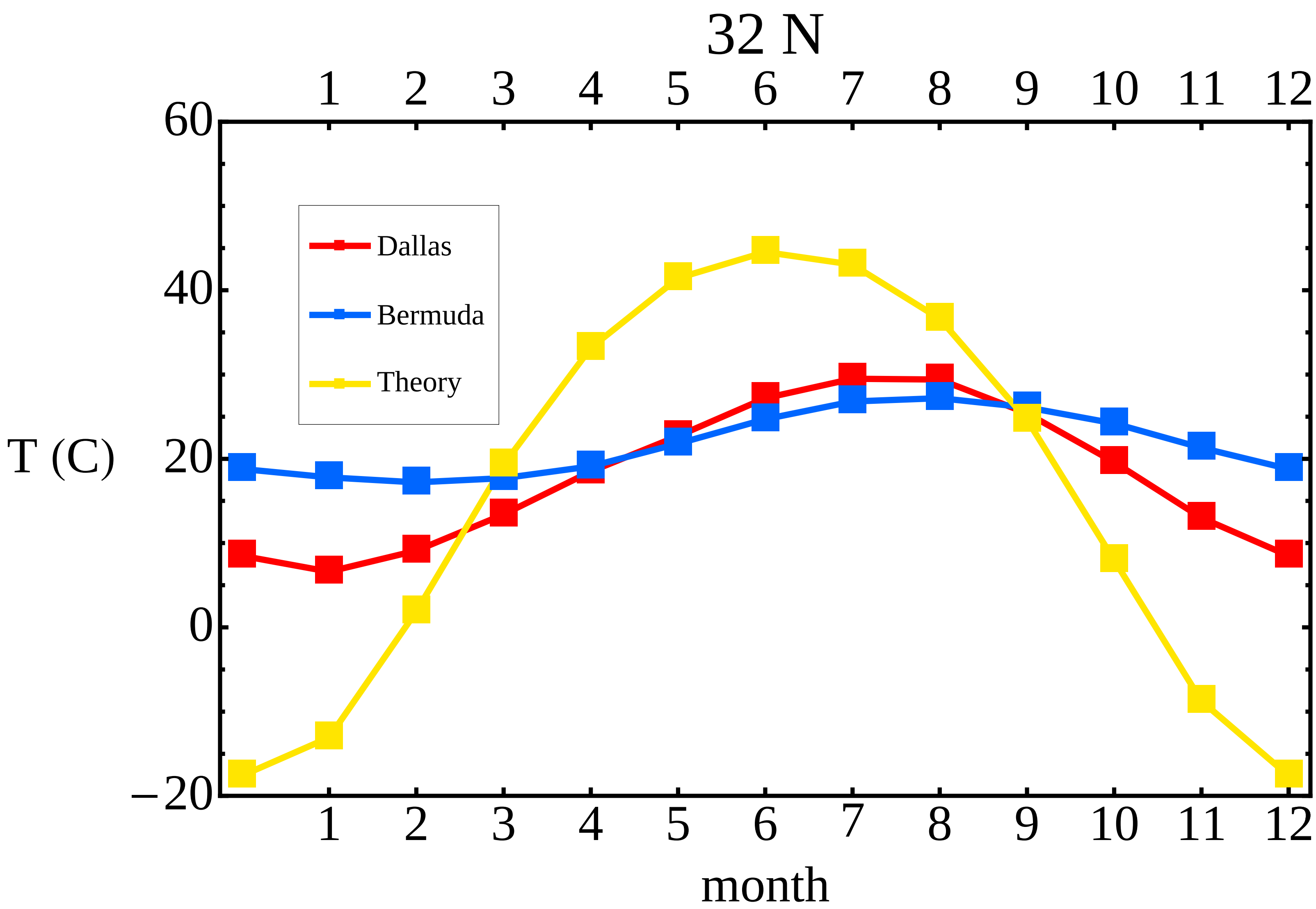
$\overline{Q}^{day}$ for 2000 A.D.



Using knowledge of how $\frac{R_o^2}{R_E^2}$ and δ vary with day of year.



Annual temperature cycles will be plotted at the selected sites.
What do you anticipate?



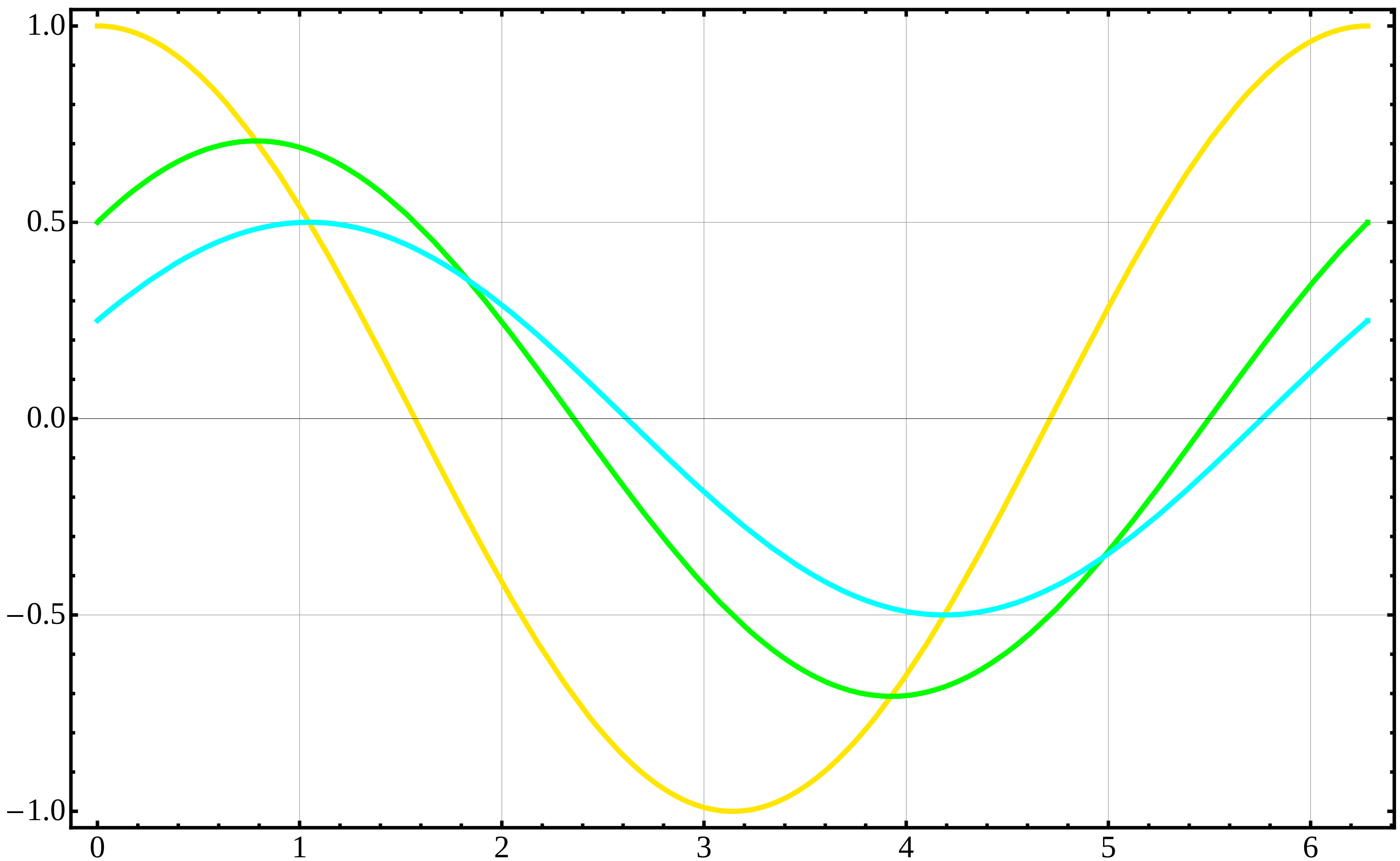
32 N
1 2 3 4 5 6 7 8 9 10 11 12
60
40
20
0
−20
T (C)
Dallas
Bermuda
Theory
month

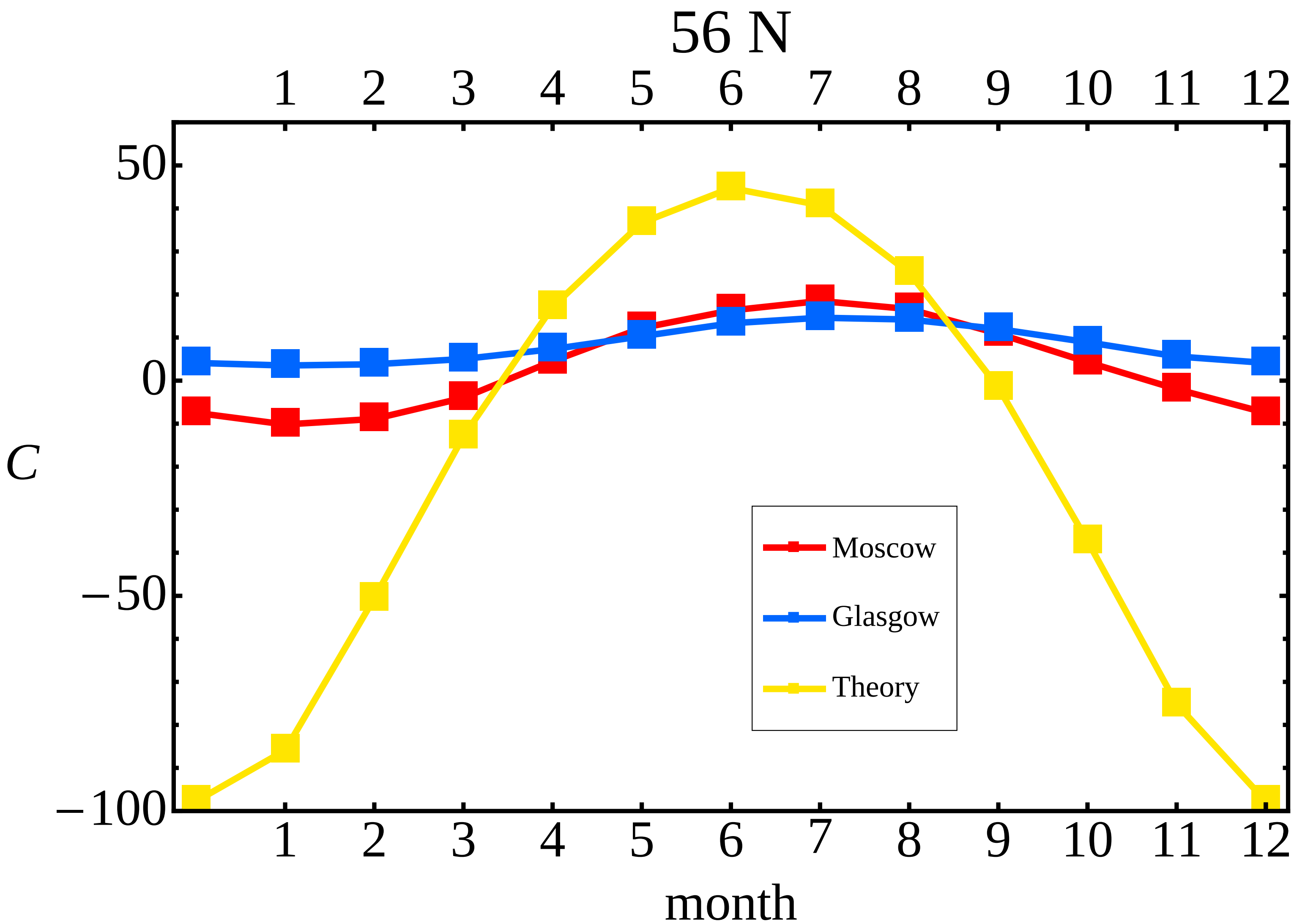
The theoretical curve is derived from our slab model:

$$\sigma T^4 = \overline{Q}^{day} \frac{1 - \alpha_p}{1 - \frac{\epsilon}{2}}$$

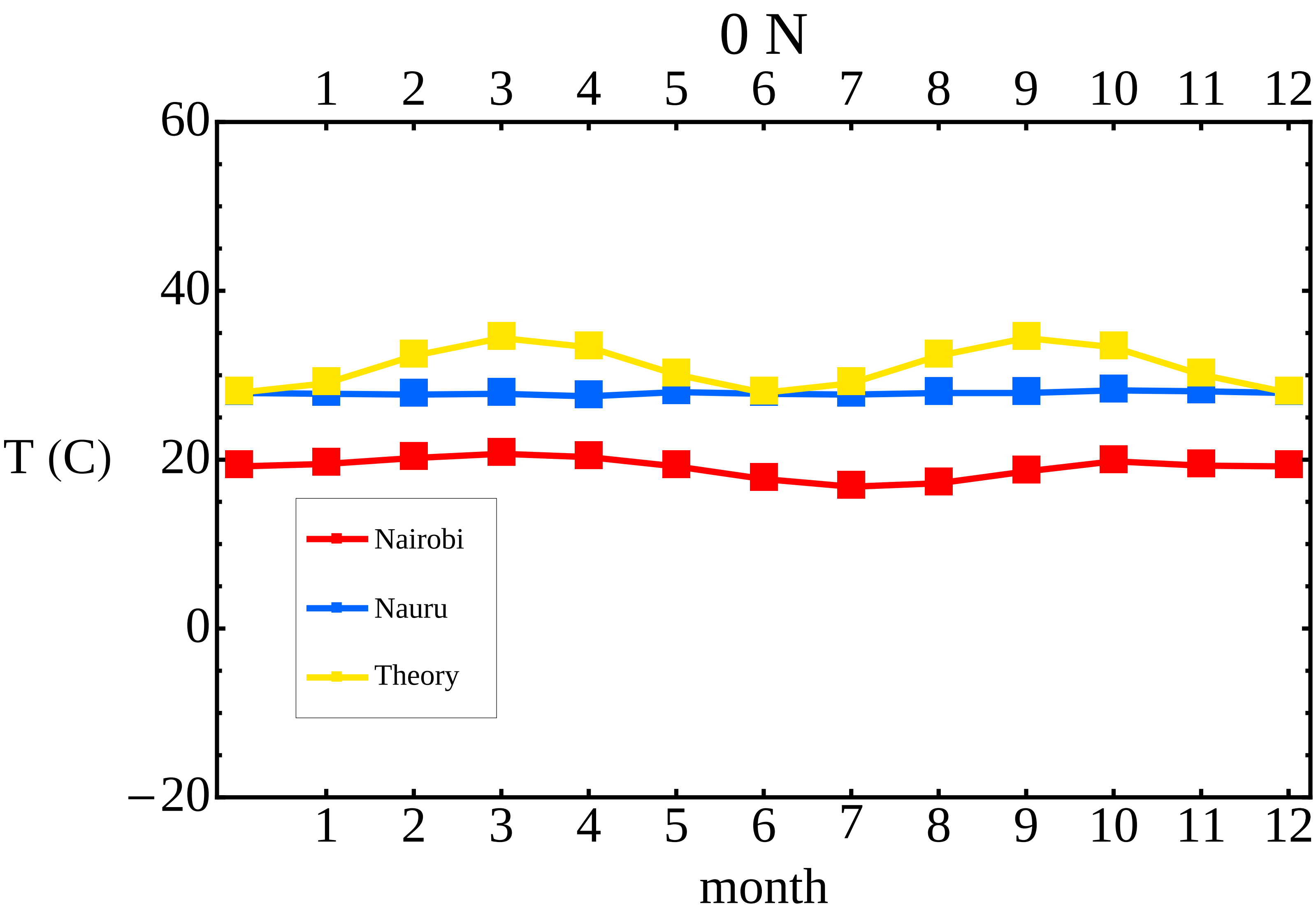
with $\alpha_p = 0.3$ and $\epsilon = 0.8$.

The model omits thermal inertia (among other things). Recall the simple result that for phase lag δ the amplitude is reduced by $\cos \delta$:





56 N
1
2
3
4
5
6
7
8
9
10
11
12
50
0
−50
−100
C
month
Moscow
Glasgow
Theory



0 N
T (C)
month
Nairobi
Nauru
Theory
60
40
20
0
−20
1
2
3
4
5
6
7
8
9
10
11
12