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METR 5223: Atmospheric Radiation

Spherical Trigonometry and Solar Radiation on a Sphere

Lecture for Spring, 2009

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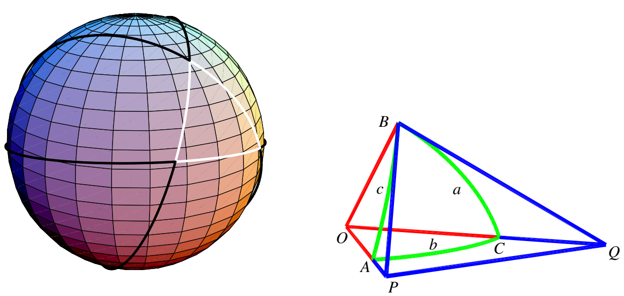

A great circle is formed by the intersection of a plane through the center of a sphere and the surface of the sphere. Three arcs of great circles can form a spherical triangle.

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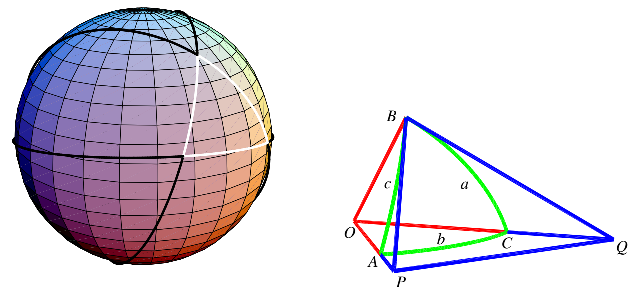

The points at the vertices of the spherical triangle are labeled A, B and C. The angles at those points have measure A, B and C. The sides opposite A, B and C have arc angle a, b and c.

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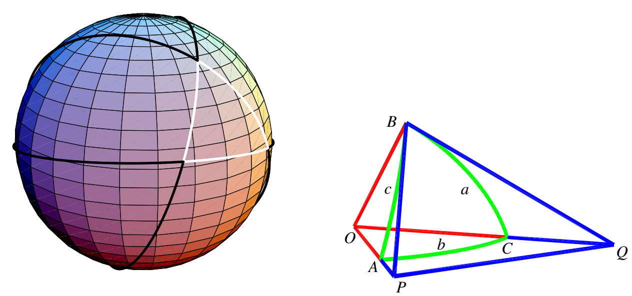

The origin, or center of the sphere, is labeled O. Point P is at the intersection of a line through O and A and a plane tangent to the sphere at B. A similar definition applies to Q.

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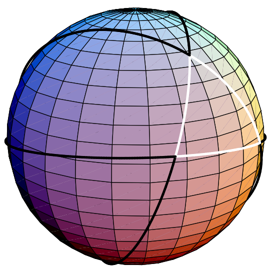
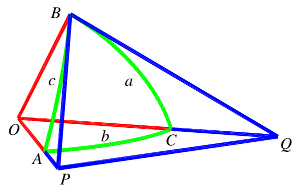


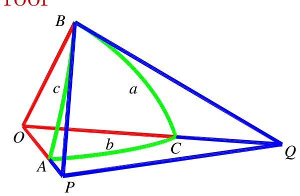


A is measured in a plane tangent to the sphere. a is the arc angle measured in a plane through the center of the sphere. A is related to a by the *law of cosines* for a spherical triangle:

$$\cos a = \cos c\ \cos b\ +\ \sin c\ \sin b\ \cos A.$$

6:

First Proof



Angle A is the angle of intersection between planes OAB and OAC. This angle is the same as the angle between the normal vectors to those two planes.

$$\frac{\overrightarrow{OA}}{|\overrightarrow{OA}|} \equiv \widehat{A} \quad \frac{\overrightarrow{OB}}{|\overrightarrow{OB}|} \equiv \widehat{B} \quad \frac{\overrightarrow{OC}}{|\overrightarrow{OA}|} \equiv \widehat{C}$$

The magnitude of $\widehat{A} \times \widehat{B}$ is $\sin c$ and the magnitude of $\widehat{A} \times \widehat{C}$ is $\sin b$.

$$\frac{\widehat{A} \times \widehat{B}}{\sin c} \cdot \frac{\widehat{A} \times \widehat{C}}{\sin b} = \cos A$$

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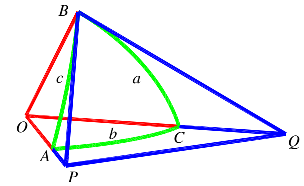


$$(\widehat{A} \times \widehat{B}) \cdot (\widehat{A} \times \widehat{C}) = \widehat{B} \cdot \widehat{C}\ \widehat{A} \cdot \widehat{A} - \widehat{B} \cdot \widehat{A}\ \widehat{A} \cdot \widehat{C}$$

$$(\widehat{A} \times \widehat{B}) \cdot (\widehat{A} \times \widehat{C}) = \cos a - \cos c\ \cos b$$

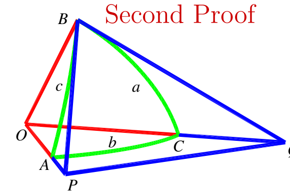
$$\cos a - \cos c\ \cos b = \cos A\ \sin c\ \sin b$$

or

$$\cos a = \cos c\ \cos b\ +\ \sin c\ \sin b\ \cos A.$$

8:

Second Proof



Applying the law of cosines for plane triangles OPQ and BPQ we have:

$$(PQ)^2 = (OQ)^2 + (OP)^2 - 2(OQ)(OP)\cos b$$

and

$$(PQ)^2 = (BQ)^2 + (BP)^2 - 2(BQ)(BP)\cos B$$

Taking the difference of these equations gives

$$\begin{aligned} 0 &= (OQ)^2 - (BQ)^2 + (OP)^2 - (BP)^2 \\ &\quad -2(OQ)(OP)\cos b + 2(BQ)(BP)\cos B \end{aligned}$$

With the Pythagorean theorem giving

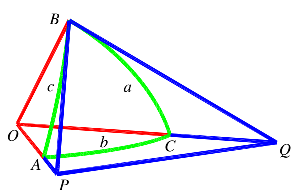
9:

$$(OQ)^2 = (BQ)^2 + (OB)^2$$
$$(OP)^2 = (BP)^2 + (OB)^2$$

$$0 = 2(OB)^2 - 2(OQ)(OP)\cos b + 2(BQ)(BP)\cos B$$

and thus

$$\cos b = \frac{OB}{OQ}\cdot\frac{OB}{OP} + \frac{BQ}{OQ}\cdot\frac{BP}{OP}\cos B$$



Noting the right triangles $\triangle OBP$ and $\triangle OBQ$,

10:

$$\cos b = \frac{OB}{OQ}\cdot\frac{OB}{OP} + \frac{BQ}{OQ}\cdot\frac{BP}{OP}\cos B$$

is written as:

$$\cos b = \cos c\ \cos a\ +\ \sin c\ \sin a\ \cos B.$$

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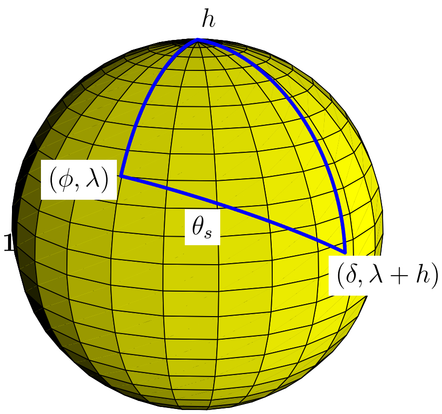

Calculate the zenith angle θ_s at a lat-lon point (ϕ, λ). The subsolar point is at latitude δ (the *solar declination angle*) and longitude $\lambda + h$ where h is the *hour angle*.

$$\cos a = \cos c\ \cos b\ +\ \sin c\ \sin b\ \cos A.$$

$$\cos\theta_s = \cos\left(\frac{\pi}{2} - \phi\right)\ \cos\left(\frac{\pi}{2} - \delta\right) + \sin\left(\frac{\pi}{2} - \phi\right)\ \sin\left(\frac{\pi}{2} - \delta\right)\ \cos h$$

$$\cos\theta_s = \sin\phi\sin\delta + \cos\phi\cos\delta\cos h$$

The distance of Earth from the sun is R_E. The mean distance is R_o. The solar flux density into a plane tangent to the sphere, but at the top of the atmosphere:

12:

$$Q = S_o\frac{R_o^2}{R_E^2}\cos\theta_s \text{ when } \cos\theta_s > 0$$

$$Q = 0 \text{ when } \cos\theta_s \le 0$$

Sunrise occurs at hour angle h_o, (and sunset occurs at hour angle $-h_o$), when $\theta_s = \frac{\pi}{2}$:

$$\sin\phi\sin\delta + \cos\phi\cos\delta\cos h_o = 0$$

The average value of Q over a day is:

$$\overline{Q}^{day} = \frac{\int_{h=\pi}^{h=-\pi} Q dh}{\int_{h=\pi}^{h=-\pi} dh}$$

Now $\int_{h=\pi}^{h=-\pi} dh = -2\pi$ and, if $\frac{R_o^2}{R_E^2}$ is nearly constant over the course of a day,

13:

$$\begin{aligned}\int_{h=\pi}^{h=-\pi} Q dh &= \int_{h=h_o}^{h=-h_o} Q dh = S_o \frac{R_o^2}{R_E^2} \int_{h=h_o}^{h=-h_o} \cos\theta_s dh \\ &= S_o \frac{R_o^2}{R_E^2} \left[h \sin\phi \sin\delta + \cos\phi \cos\delta \sin h\right]_{h=h_o}^{h=-h_o} \\ &= -2S_o \frac{R_o^2}{R_E^2} \left[h_o \sin\phi \sin\delta + \cos\phi \cos\delta \sin h\right]\end{aligned}$$

Keep it simple for a first calculation of $\overline{Q}^{day}$:

15:

1. Assume $\frac{R_o^2}{R_E^2} = 1$ exactly.
2. Know the current obliquity of Earth is 23.45°. δ oscillates between ±23.45°.
3. Calculate $\overline{Q}^{day}$ at the solstices and equinoxes.

Summary:

14:

$$\overline{Q}^{day} = \frac{S_o}{\pi} \frac{R_o^2}{R_E^2} \left(h_o \sin\phi \sin\delta + \cos\phi \cos\delta \sin h_o\right)$$

where

$$\sin\phi \sin\delta + \cos\phi \cos\delta \cos h_o = 0$$

(But sometimes θ_s is never 0 during a day, and $h_o = \pi$ or $h_o = 0$.)

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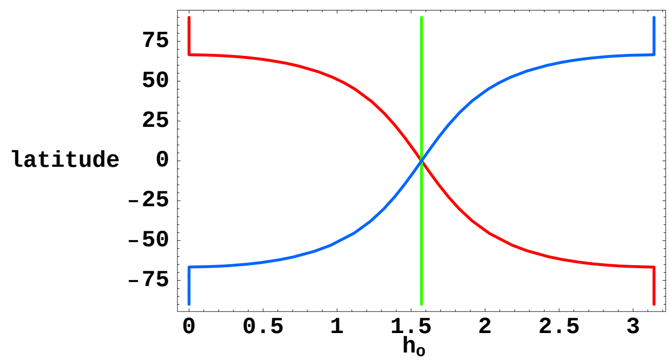

summer solstice, equinox, winter solstice

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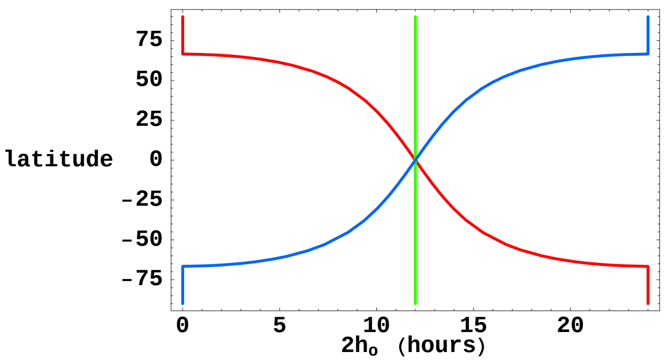

summer solstice, equinox, winter solstice

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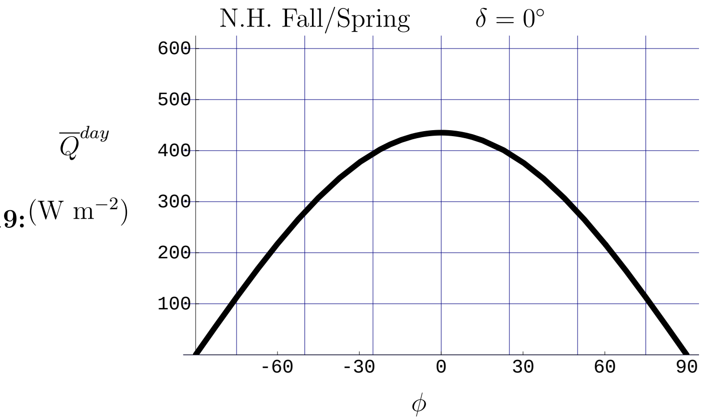

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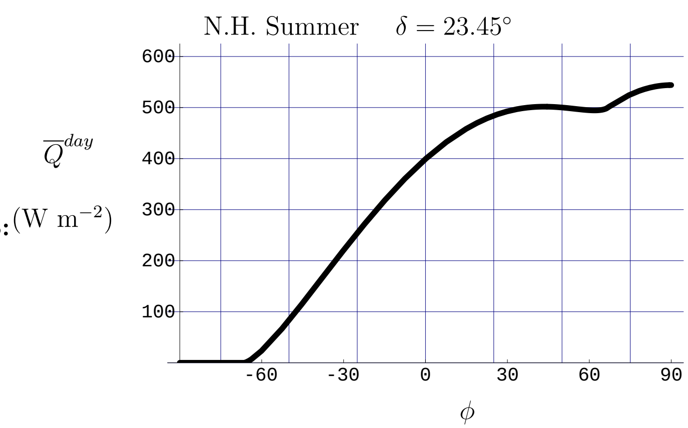

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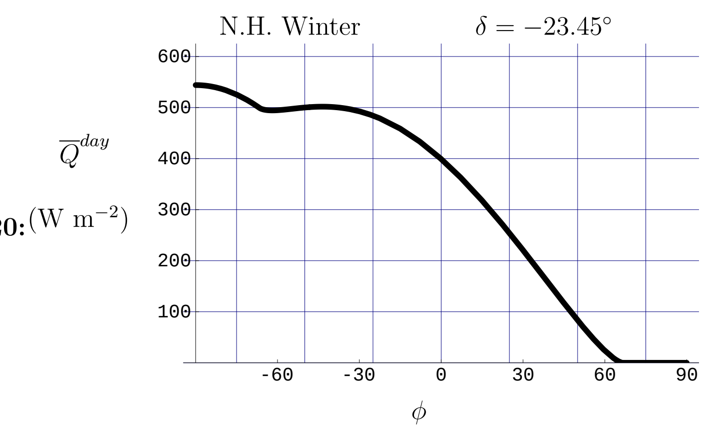

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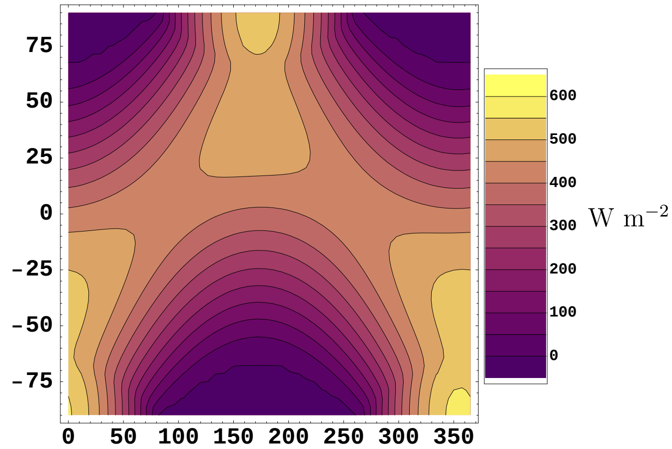
$\overline{Q}^{day}$ for 2000 A.D.



Using knowledge of how $\frac{R_o^2}{R_E^2}$ and δ vary with day of year.

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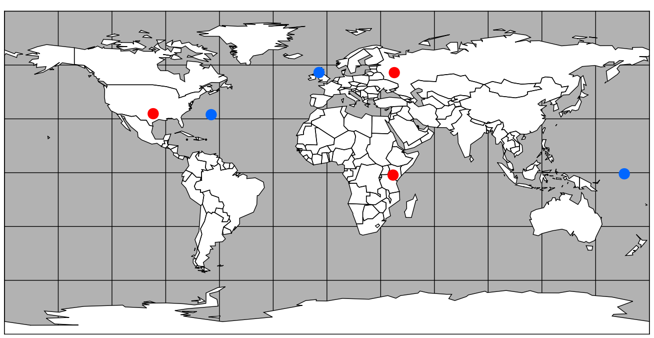

Annual temperature cycles will be plotted at the selected sites. What do you anticipate?

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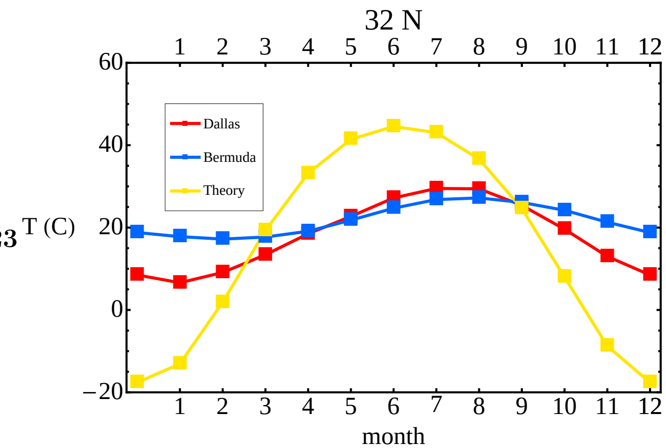

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The theoretical curve is derived from our slab model:

$$\sigma T^4 = \overline{Q}^{day} \frac{1-\alpha_p}{1-\frac{\epsilon}{2}}$$

with $\alpha_p = 0.3$ and $\epsilon = 0.8$.

The model omits thermal inertia (among other things). Recall the simple result that for phase lag δ the amplitude is reduced by $\cos\delta$:

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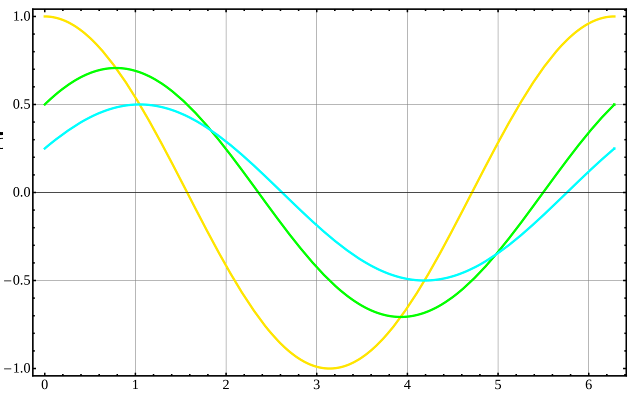

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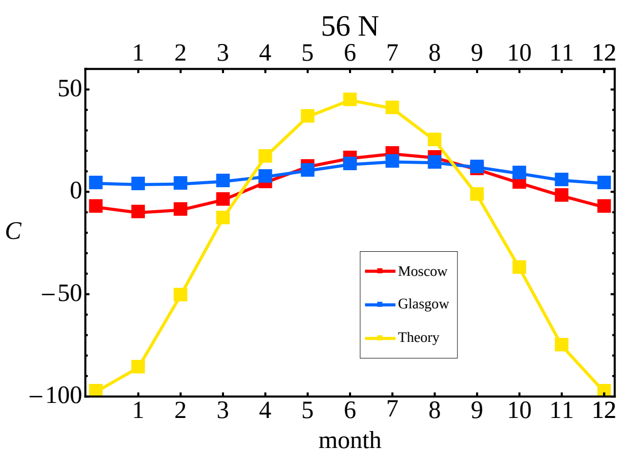

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