Well-Tempered Climate Theory

Book 1

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Abstract

J. S. Bach's *Well-tempered Clavier* consists of 48 Preludes and Fugues. My collection of approximately 48 exercises in the theory of climate needed a title. Here is what Frédéric Chopin had to say about J. S. Bach's *Well-tempered Clavier*: "Play Bach's Preludes and Fugues every day... This is the best school; no one will ever create a better... Without Bach, you cannot have freedom in the fingers, nor a clear or beautiful tone... Everything he does is perfect; it is not possible to imagine it otherwise, and the slightest change would spoil everything."

Thus, the relation of these exercises with the masterpieces of Bach is tenuous indeed. I forsee many changes in these exercises as typos and errors are discovered, and inadequate explanations induce complaints from the students. Also, the typesetting still suffers from inconsistencies. But the intent is that these exercises enjoy wide practice by those seeking to understand and participate in the debate about global warming.

The exercises offered here are distilled for the point of highlighting an important physical concept. They do not provide an authoritative calculation of a number that can be used in a policy decision. In fact, the (approximately) 24 exercises of Book 1 are so distilled that they are devoid of calculus or trigonometry. Students of Bach never had it so easy. Book 2, which is still unpublished, will offer no such reprieve.

These exercises are, or course, not a textbook. It is expected that students assigned these exercise will need plenty of supplementary material.

A Table of Useful Numerical Values appears at the end of these exercises. I am indebted to *Global Physical Climatology* by Dennis Hartmann for most of the values.

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Book 1

Climate theory without calculus

1.1 The energy of water

A mass M of ice is placed into a pan on a stove. A net power P of heating is conducted from the pan into the ice (water). The ice melts, the water warms to the boiling point, and then boils away. Find an expression (in terms of physical constants and the given parameters) for the elapsed time t_m for the ice to melt. Similarly, find the elapsed time t_w for the water to warm to the boiling point, and the elapsed time t_b for the water to boil away. Find the ratios $\frac{t_b}{t_w}$ and $\frac{t_b}{t_m}$. For the typical experience in the kitchen, the melting point of ice is $T_m = 0$ C and the boiling point is $T_b = 100$ C. Using these values, give the numerical value of the requested ratios.

Answer:

The power P is independent of time, so the heat added in time t is $\Delta Q = Pt$. The energy required to convert a mass M of ice to water is

$$\Delta Q = ML_f \tag{1.1}$$

where L_f is the latent heat of fusion. The "time to melt" t_m is therefore

$$t_m = \frac{ML_f}{P} \ . \tag{1.2}$$

The change in the temperature ΔT of the water is assumed to obey

$$\Delta Q = Mc_w \Delta T \tag{1.3}$$

where ΔQ is the heat added and c_w is the specific heat of liquid water. The "time to warm" to a boil t_w is thus

$$t_w = \frac{Mc_w(T_b - T_m)}{P} \ . \tag{1.4}$$

The energy required to convert a mass M of water to water vapor is

$$\Delta Q = ML, \tag{1.5}$$

where L is the latent heat of vaporization. The "time to boil away" t_b is

$$t_b = \frac{ML}{P} \ . \tag{1.6}$$

The ratio of these times are independent of M:

$$\frac{t_b}{t_w} = \frac{L}{c_w(T_b - T_m)} \ . \tag{1.7}$$

For the typical kitchen experience, this is

$$\frac{t_b}{t_w} = \frac{2.25 \times 10^6 \text{ J kg}^{-1}}{4.2 \times 10^3 \text{ J kg}^{-1} \text{ K}^{-1} \times 100 \text{ K}} = 5.4 \ . \tag{1.8}$$

The ratio of the time to boil to the time to melt is

$$\frac{t_b}{t_m} = \frac{L}{L_f} = \frac{2.25 \times 10^6 \text{ J kg}^{-1}}{3.34 \times 10^5 \text{ J kg}^{-1}} = 6.7 \ . \tag{1.9}$$

1.2 Sea-level pressure and the mass of the atmosphere

The average sea level pressure is $p_0 = 1013.25$ mb. Assume the base of the atmosphere is at sea level. Calculate the mass of the atmosphere. The actual mass of the atmosphere has been listed as 5.136×10^{18} kg, which we assume to be correct. It is a bit less than what you will calculate. Given that 71% of the Earth's surface is covered by oceans, use the discrepancy between the answers to *estimate* the mean height of the topography.

Answer:

The pressure p (force per unit area) at the surface is a measure of the weight of the air in a column above the surface. Consider a column with area ΔA with its base at sea level:

$$p_0 \Delta A = \Delta M g \tag{1.10}$$

Summing over all the ΔA in the surface area $A = 4\pi a^2$ of the Earth, and summing the ΔM of all the columns that contribute to the mass M of the atmosphere:

$$M = \frac{p_0 A}{g} = 5.265 \times 10^{18} \text{ kg} \tag{1.11}$$

where we have used $g = 9.81$ m s^{-2}. Our answer is greater than the listed value by $m \equiv 1.29 \times 10^{17}$ kg. To *estimate* a mean height of the topography h, we assume the elevated land masses are displacing air with density of approximately $\rho = 1$ kg m^{-3}. With land area being $0.29A$, and

$$m = 0.29Ah\rho \tag{1.12}$$

we find $h \approx 870$ m. This seems not unreasonable.

1.3 CO_2 in the atmosphere

In 1990 the concentration of CO_2 is 353 ppmv (parts per million by volume). The mass of CO_2 is listed as 2.76×10^{18} g. Are these values consistent? Suppose the 1990 mass of CO_2 in the atmosphere was consolidated into a layer of frozen CO_2 ice (so-called "dry ice") covering the surface of the Earth. How thick would the ice be?

Answer:

The universal gas law is true for any molecular weight:

$$pV = nR^*T \tag{1.13}$$

where n is the number of moles of molecules. One molecule of CO_2 occupies as much volume as one molecule of N_2. A measure of "part per million by volume", or ppmv, is really a measure of "parts per million by number". The parts per million by mass of CO_2 in the atmosphere will be the parts per million by number multiplied by the ratio of the molecular weight of CO_2 to the average molecular weight in the atmosphere. The mass of the atmosphere times this fraction gives the mass of CO_2 in the atmosphere:

$$m_{CO_2} = 5.136 \times 10^{18} \text{ kg} \times .000353 \times \frac{44.01}{28.97} = 2.75 \times 10^{15} \text{ kg} \tag{1.14}$$

which is certainly close to the listed value. If this mass was condensed into a uniform layer of dry ice onto the land surface of Earth, the volume of this ice would be Ah where A is the surface area of the Earth and h is the depth of the ice. With

$$m_{CO_2} = Ah\rho \tag{1.15}$$

where ρ is the density of dry ice, we have $h = 3.6$ mm.

1.4 Annual cycle in atmospheric CO_2

Owing to the growth of plants during the northern hemisphere growing season, CO_2 concentration typically decreases by 7 ppmv. This is sometimes said to be because "plants

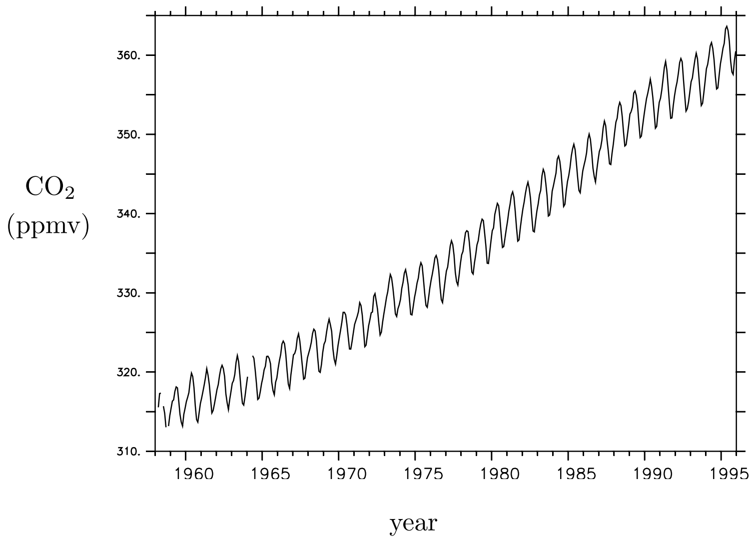

Figure 1.1: The famous measurements of atmospheric CO_2 from the top of Mauna Loa, Hawaii.

grow in summer and decay in winter". But plant material in my yard decays quite well in summer. The real reason for the annual cycle should probably be attributed to, in summer months, a greater rate of consumption of CO_2 by photosynthesis than production of CO_2 by respiration. The burning or decay of 1 kg of wood typically produces about 2 kg of CO_2, and *vice versa* for photosynthesis. Assume the density of wood is nearly that of water (meaning wood barely floats). If 7 ppmv of atmospheric CO_2 is converted to a woody material and spread evenly over all land surfaces, how thick of a layer would it be?

Answer:

As shown in Problem 1.3, 7 ppmv corresponds to a part per million by mass of:

$$7\frac{44.01}{28.97} = 10.6 \tag{1.16}$$

With the mass of the atmosphere $5.136{\times}10^{18}$ kg, the mass of CO_2 taken out in the annual cycle if therefore 5.4×10^{13}. The mass per land area is

$$5.4 \times 10^{13}\frac{1}{.29A} = 0.37 \text{ kg m}^{-2} \tag{1.17}$$

where A is the area of the Earth. Assuming the density of this woody substance is the same as water, or 1000 kg m^{-3}, the depth of the layer is 0.37 mm. Think leaf.

1.5 Radiative equilibrium with an internal heat source

A solid sphere of radius a is placed in isolation somewhere in the depths of intergalactic space. (If you don't like that notion, the sphere is placed in an evacuated room with the wall of the room maintained at zero absolute temperature.) The sphere is given an internal power supply P of heat. The surface of the sphere is assumed to radiate according to the Stefan-Boltzmann law

$$F = \epsilon\sigma T^4 \tag{1.18}$$

where σ is the Stefan-Boltzmann constant and ϵ is the emissivity, which depends on the substance. For most natural surfaces relevant to climate theory, ϵ is greater than 0.9. Here we assume that $\epsilon = 1$, which makes this surface a *perfect black body.* Such a name may seem peculiar, but the name results from the fact that if a substance is a perfect emitter of thermal radiation, Kirchoff's law requires it also to be a perfect absorber of radiation, or in a sense "black".

1. Find the equilibrium temperature T_s of the surface (assuming it is a uniform value).

2. The sphere is covered with thin layer of insulating material of thickness Δr. The material allows a heat flux through it of

$$F = -K\frac{\Delta T}{\Delta r} \tag{1.19}$$

where K is the thermal conductivity. Now find the temperature T_b immediately beneath the insulation.

Answer:
The temperature on the surface of the sphere must be sufficient to radiate away the power (or *energy flux*) P. The *energy flux density* F is in units of power per unit area. With F assumed uniform, the power radiated by the sphere is simply F times the area of the sphere. Equilibrium requires:

$$P = FA = \sigma T_s^4 4\pi a^2 \tag{1.20}$$

and

$$T_s = \left(\frac{P}{4\sigma\pi a^2}\right)^{1/4} \tag{1.21}$$

Let us pause here to note two things. First, the atmospheric sciences are notorious for stating a quantity is a *flux* even when it is really a flux per unit area, or formally a *flux density.* Secondly, just because we calculate an equilibrium temperature of a system, do we conclude that the system would tend towards the equilibrium temperature? Do we conclude that the system would often be very close to the equilibrium temperature? The first requirement for such conclusions to be true is that the equilibrium be a *stable equilibrium*, as opposed to an *unstable equilibrium.* Based on common sense, rather then mathematical analysis, the equilibrium considered here is stable: if the temperature is less than T_s the power emitted by the sphere will be less than P, energy will accumulate in the sphere, and it will be warmed. The opposite occurs for $T > T_s$. In Book 2 we will consider *unstable equilibria* in climate theory. (The equilibrium angle for a pencil to balance on its point is a familiar example of an unstable equilibrium).

Next we calculate T_b. A key assumption is the word "thin", which allows us to assume F is uniform through the depth of the insulation; spherical geometry actually requires it to increase towards the center of the sphere. We assume $\Delta r << a$ and neglect sphericity effects. With $\Delta T = T_b - T_s$, (1.19) gives

$$T_b = T_s + \frac{F \Delta r}{K} \tag{1.22}$$

(Note that $T_b > T_s$).

1.6 Radiative equilibrium with sun light

1. Consider a huge imaginary sphere centered on the sun. The radius of the sphere is equal to the average radius of the Earth's orbit R. Given the listed value of solar luminosity, find the radiation flux density onto the surface of the sphere. Compare this number with the "Solar constant" S_0.

2. Find the area of the circular shadow cast by the Earth. (Express this answer symbolically in terms of the radius of the Earth).

3. Find the power of solar radiation intercepted by the Earth. Express this answer both numerically and, for subsequent calculations, symbolically.

4. A fraction of the intercepted solar radiation is reflected back into space. This fraction is known as α_p, the *planetary albedo.* For Earth, $\alpha_p = 0.30$. Find the power of solar radiation that is absorbed by the Earth.

5. The Earth emits photons from not only the surface, but also from all layers of the atmosphere. All these layers have a variety of temperatures, so the Stefan-Boltzmann law does not strictly apply. However, it is still useful to calculate an equilibrium

radiative emission temperature T_e of the Earth. This is the temperature of an imaginary isothermal "surface", with unity emissivity, that is able to radiate away the the required flux density. Show that

$$\frac{1}{4}S_0(1-\alpha_p) = \sigma T_e^4 \tag{1.23}$$

or

$$T_e = \left[\frac{S_0(1-\alpha_p)}{4\sigma}\right]^{1/4} = 255 \text{ K} \tag{1.24}$$

6. The average distance from the sun to Venus is 1.08×10^6 km and the planetary albedo is 0.71. Find T_e for Venus.

Answer:

We will use the listed value of "Mean distance from the sun" as the radius of the imaginary sphere. The solar luminosity (the total power emitted by the sun) divided by the area of this imaginary sphere gives:

$$\frac{L_0}{4\pi R^2} = 1393 \text{ W m}^{-2} \ . \tag{1.25}$$

But the listed value $S_0 = 1367 \pm 2$ W m^{-2}. (Can you suggest reasons for the incompatibility?) We will use the listed value of S_0 in subsequent calculations.

The area of Earth's shadow is quite simply πa^2 where a is the radius of the Earth. The power of solar radiation intercepted by Earth is:

$$S_0 \pi a^2 = 1.74 \times 10^{17} \text{ W} \ . \tag{1.26}$$

The fraction absorbed is one minus the fraction reflected, or $1 - \alpha_p$. Therefore the power absorbed is

$$(1-\alpha_p)S_0\pi a^2 \ . \tag{1.27}$$

Assume the global energy budget of earth is in radiative equilibrium. By this we mean the production of energy by radioactive decay of minerals is negligible, and the temperature of the solid earth, oceans and atmosphere is not changing (global warming ignored!). In this pure state of radiative equilibrium, the power emitted by Earth must be the power of solar radiation absorbed. Using the concept of radiative emmission temperature T_e, the "balance of power" is

$$(1-\alpha_p)S_0\pi a^2 = 4\pi a^2 \sigma T_e^4 \tag{1.28}$$

and (1.24) follows, with value $T_e = 255$ K. The average surface temperature of Earth is alleged to be 288 K. Temperatures of 255 K are found higher in the atmosphere. On average, that is where photons originate from that are contributing to the outgoing radiative flux.

The calculation of T_e for Venus is similar. For the orbital radius of Venus, $S_0 = 2637$ W m^{-2}, and $T_e = 240$ K. The observed surface temperature of Venus is 700 K. We can safely conclude that very little of the infrared flux from the surface goes into space without absorption. The outward flux of radiation at the top of the atmosphere of Venus originates from layers that are much higher and cooler than the surface.

In problem 1.11 we will explore a model that can predict the difference between the observed average surface temperatures and T_e. The reason for the difference is a form of "radiative insulation" known as the *greenhouse effect*. The analysis of the greenhouse effect is somewhat different from the "conductive insulation" that was modeled in Problem 1.5. Before we consider the greenhouse effect, we turn to a very useful theory for the *local* equilibrium temperature of a patch of the Earth's surface. This theory is useful in its own right, but its presentation here allows a timely introduction of concepts used in the analysis of the *global* greenhouse effect.

1.7 Energy balance at the surface

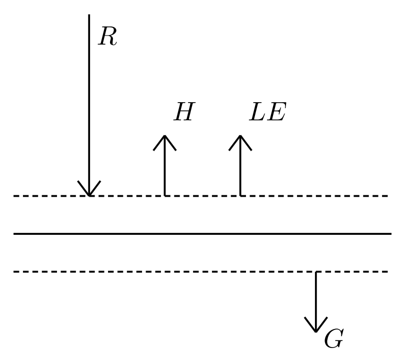

Figure 1.2: Energy balance at surface. R is net absorbed radiation, H is the sensible heat flux, LE is the latent heat flux. G is the heat flux into the ground.

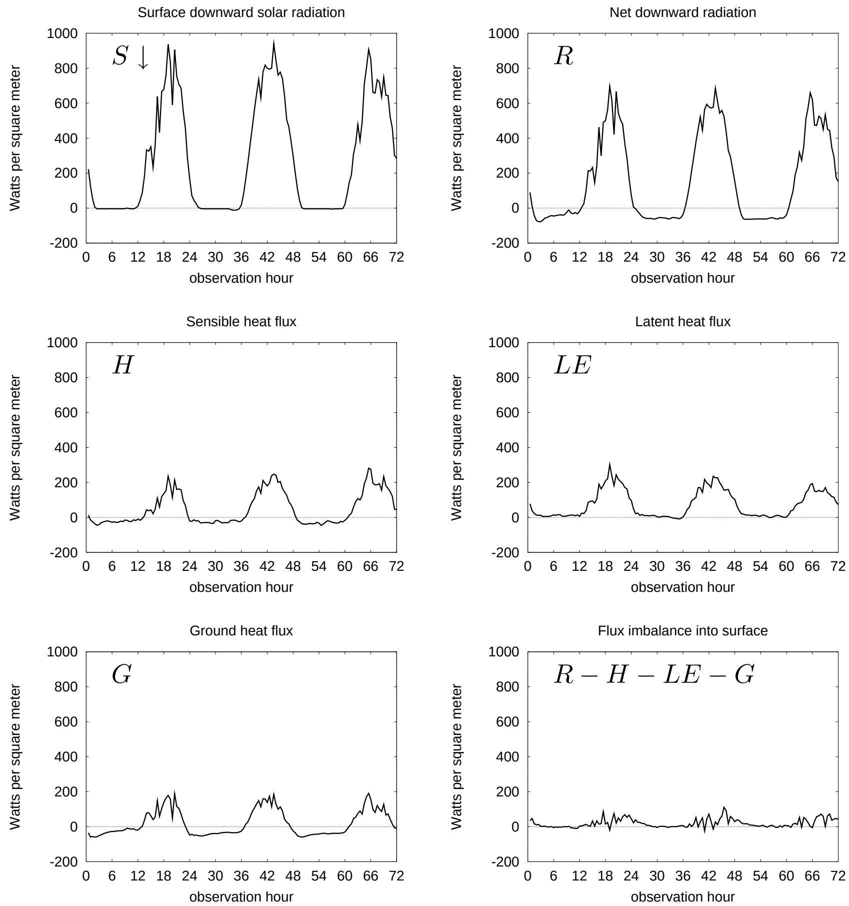

Figure 1.3: Observations of surface energy fluxes from 0 Z, June 7, 1999 at the Norman, Oklahoma mesonet site. Notice that, unlike the theory, $R - H - LE - G \neq 0$. This is due to limitations of errors in the observations, rather than a fault in the conservation of energy principle.

Consider a very thin volume enveloping the top few molecules of a land surface and the lowest few molecules of the atmosphere, as in Figure 1.51. Imagine the limit of a very thin pizza box. The area of either the top or bottom of the box is denoted A. Energy added into this volume can either warm the mass (what little there is) or evaporate water. Writing the evaporative mass flux density from the top of the volume as E, the mass of water evaporated in time interval Δt is $\Delta t AE$. The energy required to produce this evaporation is $ALE\Delta t$. The net amount of heat into the box during this time is $A(R - H - G)\Delta t$ where the term in parentheses is the net flux density into the volume. Note the sign conventions in the following: R is the net radiation **in** from above, H is the heat flux **out** into the atmosphere, and G is the conductive flux of heat **out** into the ground. Conservation of energy within this volume is:

$$C\Delta T + ALE\Delta t = A(R - H - G)\Delta t \tag{1.29}$$

where C is a vaguely defined heat capacity (something like average specific heat times the mass). In the mathematical limit of a thin surface, there are no lateral fluxes to account for. With C being infinitesimal, there is negligible heat that can be stored in the thin surface. There, for almost any practical calculations:

$$R = H + LE + G \ . \tag{1.30}$$

With E the evaporative mass flux density **out** into the atmosphere, LE is the rate of energy supplied by the volume to effect this evaporation. Thus (1.30) is simply an expression of "energy flux in equals energy flux out". An example of observations of this balance are shown in Figure 1.3.

The net radiative flux density into the surface can be partitioned into four streams, where S is short-wave (solar) radiation, and F is long-wave (infrared) radiation. The arrows define which way the photons are moving that are included in the streams, so all these arrow-quantities are non-negative:

$$R = S\downarrow - S\uparrow + F\downarrow - F\uparrow \ . \tag{1.31}$$

The surface emits negligible radiation in the short-wave portion of the spectrum, so $S\uparrow$ is purely the reflected portion:

$$S\uparrow = \alpha_s S\downarrow \tag{1.32}$$

where α_s is the surface albedo, sometimes called the *reflectivity for short-wave radiation.* The surface emits long-wave radiation according to the Stefan-Boltzmann Law and reflects long-wave radiation according to Kirchoff's Law:

$$F\uparrow = \epsilon\sigma T_s^4 + F\downarrow (1 - \epsilon) \ . \tag{1.33}$$

The reflectivity is, by definition, equal to one minus the absorptivity. Futhermore, we have also assumed that the absorptivity is equal to the emissivity, so $(1 - \epsilon)$ is actually the reflectivity. Such an assumption is always true when considering light of certain wavelength. At a certain wavelength, the absorptivity is equal to the emissivity by Kirchoff's Law, which is a quite amazing fact about physics. ("Good absorbers are good emitters" may be a way to remember Kirchoff's law.) But here the incoming spectrum of radiation is slightly different from the outgoing spectrum. But because the wavelength distribution is close to being the same, the effective emissivity ϵ for upwards thermal radiation will also be used for the absorptivity for downwards thermal radiation. Thus,

$$R = S\downarrow(1-\alpha_s) + \epsilon(F\downarrow - \sigma T_s^4)\ . \tag{1.34}$$

The heat flux density H and the evaporative flux density E from the land (or water) surface into the atmosphere can be rather simple to model in an average sense (a spatial and/or time average). Despite all the complexity of the process, the average heat flux density can often be modeled accurately with:

$$H = c_p \rho C_H U_r \left[T_s - T_a(z_r)\right] \tag{1.35}$$

where ρ is the density of the air, C_H in an aerodynamic heat transfer coefficient that depends on the roughness of the surface, T_s is the skin temperature of the surface and $T_a(z_r)$ is the temperature of the air at the "reference height" for the calibration of the formula (meaning the height for which the value of C_H will be valid). Likewise, U_r is the average windspeed at the reference height. It is a pleasant fact the C_H can be calibrated from laboratory experiments. In the atmospheric sciences, values of C_H are usually tabulated for use with $z_r = 10$ m. Over a water surface, or an unvegetated soil surface, (1.35) is relatively straight foward to apply, provided that T_s can be either measured or otherwise specified.

Similarly, the evaporative flux is given by

$$E = \rho C_E U_r \left[q_s - q_a(z_r)\right] \tag{1.36}$$

where q_s is the specific humidity (mass of water vapor per mass of air) of the "skin" of air immediately adjacent to the land or water surface, and q_a is the specific humidity at the reference level. Over a water surface, q_s is easy to know: it is just the saturation specific humidity for the temperature of the water surface. Over other surfaces, such as vegetation, this quantity is not so easy to know, and neither is C_E.

1. Nevertheless, for an ocean surface, or a dry unvegetated soil, the transfer formulas are easy to apply. For example, consider a dry sand with $\alpha_s = 0.35$ and $\epsilon = 0.9$. A broken clock is exactly right twice a day. Likewise, twice a day, in a typical diurnal cycle, $G = 0$ (usually early morning and late afternoon). Assume $S\downarrow = 400$ W m^{-2}, $G = 0$, $F\downarrow = 300$ W m^{-2}, $T_a = 20$ C, $C_H = 2\times10^{-3}$, $\rho = 1.2$ kg m^{-3}, and $U_r = 10$ m s^{-1}. Find T_s. The sand is dry so $q_s = q_a(z_r)$ and $E = 0$.

2. Also try the calculation with $U_r = 1$ m s^{-1}.

3. Also try the calculation with $U_r = 1$ m s^{-1} with $\epsilon = 1$.

Answer:
You will need to solve $R - H = 0$ for T_s:

$$S \downarrow (1 - \alpha_s) + \epsilon(F \downarrow - \sigma T_s^4) - c_p \rho C_H U_r (T_s - T_a) = 0 \tag{1.37}$$

Everything is given in this equation except for T_s, which can be found by an iterative procedure of your choice. The answer is $T_s = 25.2$ C, which is 5.2 C greater than the air temperature. With $U_r = 1$ m s^{-1}, $T_s = 38.97$ C. Also changing to $\epsilon = 1$, $T_s = 36.39$ C. Thus our calculations seem to tolerate errors, or lack of knowledge, of the precise value of ϵ when compared with the big changes in T_s caused by the the natural variability in the wind speed. If you are contemplating walking across the pavement barefoot, you should be more concerned about knowledge of the wind speed, rather than the emissivity.

1.8 The atmospheric boundary layer

The temperature of the bottom 1/10 th of the atmosphere increases by 15 K between 6 CST and 18 CST. Winds are light. No fronts or storms have passed over the area. Assume the atmosphere exchanges negligible radiation with the ground (not likely). What was the average net sensible heat flux H from the ground into the atmosphere during this time?

Answer:
The bottom 1/10'th of atmosphere is assumed here to mean 10% of the mass of the atmospheric column, or 1033 kg m^{-2}. The change in temperature is assumed to be entirely from the sensible heat flux from below. For an arbitrary area A of the column,

$$\frac{\Delta Q}{A} = H \Delta t = \frac{M}{A} c_p \Delta T \tag{1.38}$$

and

$$H = \frac{M}{A} c_p \frac{\Delta T}{\Delta t} = 1033 \text{ kg m}^{-2} \times 1004 \text{ J kg}^{-1} \text{ K}^{-1} \times 15 \text{ K} \div 43200 \text{ s} = 360 \text{ W m}^{-2} \tag{1.39}$$

1.9 The oceanic mixed layer

1. The temperature of the top 40 meters of ocean increases by 10 K over six months. What is the average net heat flux density into the ocean surface that could cause this warming?

2. 50 cm of water evaporates from the ocean over six months. What is the average latent heat flux density into the atmosphere?

3. 100 W m^{-2} of sensible heat flux density goes from the ocean into the atmosphere during the six months. Using also the answers in (a) and (b), what was the net radiative flux density absorbed by the ocean?

Answer:

G is diagnosed from the change in energy per unit area within the column.

$$G = \rho_w h c_w \Delta T \frac{1}{\Delta t} = 107 \text{ W m}^{-2} \tag{1.40}$$

where h is 40 m. The evaporative mass flux density is simply:

$$E = 50 \text{ cm} \times \rho_w \frac{1}{\Delta t} = 3.17 \times 10^{-5} \ \text{ kg m}^{-2} \text{ s}^{-1} \tag{1.41}$$

With $L = 2.5 \times 10^6$ J kg^{-1},

$$LE = 79 \text{ W m}^{-2} \tag{1.42}$$

With H given, energy balance yields R:

$$R = H + LE + G = 286 \text{ W m}^{-2} \tag{1.43}$$

1.10 Heat capacity of the oceans versus the atmosphere

At what depth in the ocean does a column of water above have the same mass as the column of atmosphere above the water? At what depth in the ocean does a column of water above have the same heat capacity as the column of atmosphere above the water? (To answer this last question, consider for what depth an identical heat input ΔQ into both columns would produce the same temperature change ΔT).

Answer:

Let ΔA be the area of the column. As in Problem (1.2), the mass of atmosphere in a column is $p_0 \Delta A / g$ where p_0 is the surface pressure. The mass of water in a column of depth h is $h \Delta A \rho_w$ where ρ_w is the density of water. The masses are the same when

$$h = \frac{p_0}{\rho_w g} = \frac{1.013 \times 10^5 \text{ kg m}^{-1} \text{ s}^{-2}}{1.00 \times 10^3 \text{ kg m}^{-3} \times 9.81 \text{ m s}^{-2}} = 10.3 \text{ m} \tag{1.44}$$

If the heat input is distributed uniformly in a column of atmosphere, so that ΔT is the same value everywhere, then

$$\Delta Q = \frac{p_0 \Delta A}{g} c_p \Delta T \tag{1.45}$$

Similarly, in the column of ocean,

$$\Delta Q = \Delta A h \rho_w c_w \Delta T \tag{1.46}$$

For both the column of ocean and atmosphere to have the same response ΔT for the same input ΔQ:

$$\frac{p\Delta A}{g} c_p = \Delta A h \rho_w c_w \tag{1.47}$$

or

$$h = \frac{p c_p}{g c_w \rho_w} = 2.48 \text{ m} \tag{1.48}$$

Salinity effects have been neglected.

1.11 Planet X with a greenhouse effect

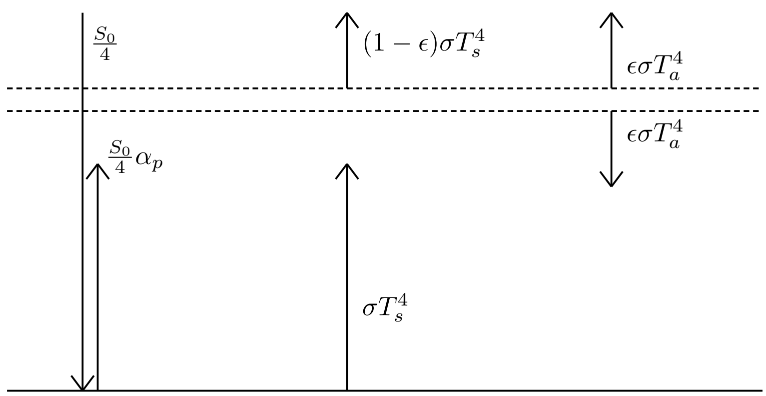

Figure 1.4: Radiation streams in a pure radiative equilibrium of a Planet X with a "greenhouse effect". The atmosphere does not absorb any solar radiation. A fraction α_p of solar radiation is reflected back into space by the combined effect of the surface and the atmosphere.

Consider Figure 1.4, which depicts the radiation streams of a fictional Planet X. The radiation streams are time-averaged and space-averaged, and so have a constant value. Even though Planet X is really a rotating sphere, a plane-parallel picture is useful to help us write down the equations for energy balance.

Planet X differs from Earth in that the atmosphere of Planet X does not absorb any solar radiation. Furthermore, the atmosphere of Planet X is assumed isothermal with temperature

T_a. Therefore, in modeling the radiation, the atmosphere can be modeled as an isothermal plane of glass that absorbs and emits only longwave radiation. Planet X is similar to Earth in that solar radiation is reflected back into space by both the atmosphere and the surface. The energy budget analysis does not require knowledge of the fraction reflected by each, only knowledge of the total amount reflected is required. Also, like Earth, the time-averaged flux of heat into the ground G is approximately zero in the analysis of climate equilibrium. But unlike Earth, Planet X has neither a sensible heat flux nor a latent heat flux. The transport of energy is entirely by radiation.

The thermal radiative flux density out of the top of the atmosphere is

$$F \uparrow = \epsilon\sigma T_a^4 + (1-\epsilon)\sigma T_s^4 \ . \tag{1.49}$$

An equilibrium condition of zero net radiation leaving the top of the atmosphere requires

$$-\frac{1}{4}S_0(1-\alpha_p) + \epsilon\sigma T_a^4 + (1-\epsilon)\sigma T_s^4 = 0 \ . \tag{1.50}$$

An equilibrium condition of zero net radiation entering the surface requires

$$\frac{1}{4}S_0(1-\alpha_p) + \epsilon\sigma T_a^4 - \sigma T_s^4 = 0 \ . \tag{1.51}$$

Adding these conditions is one way to arrive at the condition for energy equilibrium of the atmosphere:

$$2\epsilon\sigma T_a^4 - \epsilon\sigma T_s^4 = 0 \tag{1.52}$$

1. This simple model of Planet X can be used for both equilibrium and non-equilibrium calculations. An example of a non-equilibrium calculation, assume T_s and T_a remain at fixed values while the emissivity of the atmosphere increases from ϵ_1 to ϵ_2. Show that $F \uparrow$ decreases if $T_s > T_a$.

2. Show that, assuming equilibrium conditions,

$$\frac{S_0(1-\alpha_p)}{4} = (1-\epsilon/2)\sigma T_s^4 \tag{1.53}$$

and

$$T_s = \left[\frac{S_0(1-\alpha_p)}{4\sigma}\frac{1}{1-\epsilon/2}\right]^{1/4} \ . \tag{1.54}$$

The model "saturates" when ϵ reaches 1. Show that a maximum in surface temperature of $1.18T_e$ is reached at $\epsilon = 1$. (With the limitation of the model having an isothermal atmosphere, the model cannot be used to predict the very large amplification of surface temperature that occurs on a planet like Venus).

3. Modify the model to include absorption of solar radiation in the atmosphere, with rate $k\frac{S_0(1-\alpha_p)}{4}$. Show that

$$T_s = \left[\frac{S_0(1-\alpha_p)}{4\sigma}\frac{1-k/2}{1-\epsilon/2}\right]^{1/4} \tag{1.55}$$

Answer:

Let $\Delta F\uparrow$ be the difference between $F\uparrow$ with ϵ_2 minus $F\uparrow$ with ϵ_1. Assume $\epsilon_2 > \epsilon_1$. From (1.49),

$$\Delta F\uparrow = \sigma\left(\epsilon_2 - \epsilon_1\right)\left(T_a^4 - T_s^4\right) < 0. \tag{1.56}$$

That calculation demonstrates the fundamental mechanism of the greenhouse effect: as ϵ increases, the upward flux density of thermal radiation at the top of the atmosphere shifts towards that for cooler atmospheric temperatures and away from that for warmer surface temperatures.

Using (1.52) in either (1.51) and (1.50) gives

$$\frac{1}{4}S_0(1-\alpha_p) = \left(1-\frac{\epsilon}{2}\right)\sigma T_s^4 \tag{1.57}$$

Note the consistency of this calculation with Problem 1.6 in the limit of $\epsilon = 0$: $T_s = T_e$. In the limit of $\epsilon = 1$, $T_s = 2^{1/4}T_e$. Applied to Earth with $T_e = 255$ K, this limit predicts $T_s = 303$ K, a bit greater than the observed $T_s = 288$ K. We might be attempted to find a value of $\epsilon < 1$ so that the model would exactly right for Earth. But it would probably be right for the wrong reason, given all the approximations in the model that are not really suitable for Earth. Nevertheless, the model gives us confidence that a greenhouse effect, similar to that on Planet X, is what is responsible for T_s being 33 K greater than T_e on Earth.

Including atmospheric absorption of solar radiation, atmospheric energy balance (1.52) is replaced by

$$2\epsilon\sigma T_a^4 - \epsilon\sigma T_s^4 - k\frac{S_0(1-\alpha_p)}{4} = 0 \tag{1.58}$$

Substitution into either (1.51) or (1.50) gives

$$\frac{1}{4}S_0(1-\alpha_p)\left(1-k/2\right) = \left(1-\epsilon/2\right)\sigma T_s^4 \tag{1.59}$$

and (1.55) follows. Note if $k = \epsilon$, the greenhouse effect is null, and $T_s = T_e$. This fact was the basis for the "nuclear winter" doomsday scenario that received a lot of publicity in the 1980's. A nuclear winter is theorized to result from smoke and debris from a nuclear war increasing k, but not necessarily α_p. Only with $k < \epsilon$, as is appropriate for modeling Earth, is $T_s > T_e$.

1.12 Planet Earth with a greenhouse effect

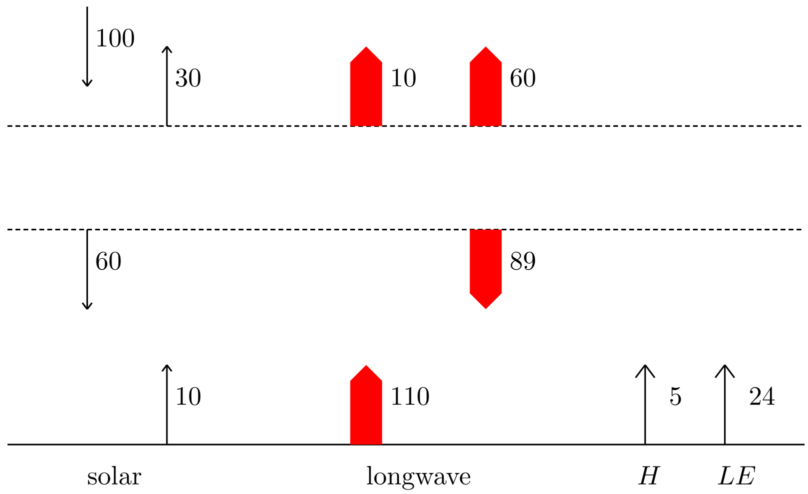

Figure 1.5: Average radiation streams in the Earth's atmosphere. The schematic shows an atmosphere bounded by a dashed lines, a gap is depicted at the Earths surface for clarity of the labeling. The units are in percent of the magnitude of the average inward solar radiation flux density at the top of the atmosphere, which would be $S_0/4$.

Figure 1.5 shows energy fluxes at the surface and top of the atmosphere as deduced from observations. In contrast to Figure 1.4, the thermal radiation emitted upwards by the atmosphere is less than the radiation emitted downwards towards the ground by the atmosphere. This is because the atmosphere is not isothermal, and the radiation out of the top is, on average, coming from layers cooler than the radiation coming onto the ground.

1. Show that the figure depicts zero net input of energy into the atmosphere.
2. Show that the figure depicts surface energy balance with $G = 0$.
3. Show that the figure indicates that 20 units of solar radiation is absorbed in the atmosphere.
4. Let $\langle\ \rangle$ be a global averaging operator. It also may include a time averaging operation, say over a year. The upwards surface flux in the figure is being shown for global-average conditions. Assuming the surface emissivity is unity, then at the surface:

$$\langle F \uparrow \rangle = \left\langle \sigma T_s^4 \right\rangle \tag{1.60}$$

Find $\langle T_s^4 \rangle / T_e^4$. Assuming $\langle T_s^4 \rangle = \langle T_s \rangle^4$, find $\langle T_s \rangle / \langle T_e \rangle$, using the numbers in the figure. If $T_e = 255$ K, give the value of $\langle T_s \rangle$.

5. Now suppose over half the globe $T_s = \langle T_s \rangle + \delta$, and over the other half $T_s = \langle T_s \rangle - \delta$. Assuming $\delta = 15K$, now find $\langle T_s \rangle / \langle T_e \rangle$, using the numbers in the figure. If $T_e = 255$ K, give the value of $\langle T_s \rangle$.

Answer:

The first three answers are obtained rather "easily", provided some physical facts are understood. First, it assumed that there is an equilibrium amount of water vapor in the atmosphere, so the rate of condensation in the atmosphere is equal to the rate of evaporation from the surface. All energy used for evaporation is returned as thermal energy when water vapor condenses. However, there are some subtleties to the budget analysis, which perhaps are best left alone here: the latent heat of fusion L decreases with increasing temperature and will generally not be the same value aloft as it was at the surface.

The last questions are solved with

$$\frac{\langle \sigma T_s^4 \rangle}{\sigma T_e^4} = \frac{110}{70} \equiv r \ . \tag{1.61}$$

With the assumption $\langle T_s^4 \rangle = \langle T_s \rangle^4$,

$$\langle T_s \rangle = r^{1/4} T_e = 1.120 T_e = 285.5 \text{ K} \tag{1.62}$$

If instead we consider a more realistic distribution of T_s, with cooler poles and a warmer equator,

$$\begin{aligned} \langle T_s^4 \rangle &= \frac{1}{2} \left(\langle T_s \rangle + \delta \right)^4 + \frac{1}{2} \left(\langle T_s \rangle - \delta \right)^4 \\ &= \langle T_s \rangle^4 + 6 \langle T_s \rangle^2 \delta^2 + \delta^4 = r \ T_e^4 \end{aligned} \tag{1.63}$$

As a quadratic equation, this has a solution of

$$\langle T_s \rangle^2 = \frac{1}{2} \left[-6\delta^2 + \sqrt{32\delta^4 + 4rT_e^4} \right] = (284.3 \text{ K})^2 \tag{1.64}$$

Note the simple assumption of uniform T_s gives a good estimate for most purposes. It predicts a greenhouse effect of 30.5 K, rather than 29.3 K for the more realistic model.

1.13 Storms, bombs and meteors

1. A thunderstorm dumps 10 cm of rain over 1000 km^2. How many Joules of latent heat were released? The mightiest H-Bomb ever exploded was "Tsar Bomba", which

released 4.2×10^{17} J. By definition, 1 Megaton of TNT equivalent $\equiv 4.186 \times 10^{15}$ J. This unit of energy is often referred to as simply "megaton", and sometimes abbreviated MT. "Tsar Bomba" was a 100 MT bomb. The Hiroshima bomb was a"mere" 0.01 MT, or 10 Kiloton. Also express your answer in units of "Tsar Bombas".

2. Suppose a spherical meteor smashes into the Earth with a speed of 40,000 mph. The density of the meteor is that of typical rock: $5,000$ kg m^{-3}. What diameter of meteor would release the same amount of energy as the thunderstorm?

3. The rain moistens the soil, leaves puddles, and causes the plants to transpire more water into the atmosphere. Of the average daily budget of $R = 300$ W m^{-2}, one-third is converted into a latent heat flux LE. How many days are required to evaporate the equivalent of 10 cm of rain?

Answer:

The volume of rain produced by the thunderstorm is

$$V = 10 \text{ cm} \times 1000 \text{ km}^2 \times \frac{0.01 \text{ m}}{\text{cm}} \times \left(\frac{1000 \text{ m}}{\text{km}}\right)^2 = 1.0 \times 10^8 \text{ m}^3 \tag{1.65}$$

The mass of rain is:

$$M = \rho_w V \tag{1.66}$$

where $\rho_w = 1000$ kg m^{-3} is the density of liquid water. So $M = 1.0 \times 10^{11}$ kg. The energy released is

$$Q = LM \tag{1.67}$$

where $L = 2.5 \times 10^6$ J kg^{-1} is the latent heat of vaporization for water. So $Q = 2.5 \times 10^{17}$ J. In units of "Tsar Bombas" this is:

$$Q = 2.5 \times 10^{17} \text{ J} \times \frac{1 \text{ Tsar Bomba}}{4.2 \times 10^{17} \text{ J}} = 0.6 \text{ Tsar Bombas} \tag{1.68}$$

Assume the energy "released" by the meteor is its kinetic energy prior to impact:

$$Q = \frac{1}{2} m v^2 \tag{1.69}$$

The speed v is:

$$v = 40,000 \frac{\text{mi}}{\text{hr}} \times \frac{1609 \text{ m}}{\text{mi}} \times \frac{1 \text{ hr}}{3600 \text{ s}} = 17,900 \text{ m s}^{-1} \tag{1.70}$$

The mass is the density ρ times the volume of the spherical meteor:

$$m = \rho \frac{4}{3} \pi R^3 \tag{1.71}$$

where R is the radius of the sphere. So

$$R = \left(\frac{3Q}{2\pi \rho v^2} \right)^{1/3} = 42.1 \text{ m} \tag{1.72}$$

The diameter is $2R$, or 84.2 m.

10 cm of water has a mass of 100 kg m^{-2}, and this value must be equal to Et, where t is the elapsed time that we are seeking. But $LE = 100$ W m^{-2}.

$$\begin{aligned} t &= \frac{1}{E} \times 100 \text{ kg m}^{-2} \\ &= \frac{L}{100 \text{ W m}^{-2}} \times 100 \text{ kg m}^{-2} \\ &= 2.5 \times 10^6 \text{ s} = 29 \text{ days} \end{aligned} \tag{1.73}$$

1.14 Roasting the dinosaurs

A television program on the Discovery Channel stated "The asteroid impact that killed the dinosaurs was caused by an asteroid six miles wide impacting at 40,000 mph".

1. Is the speed estimate reasonable? Is it consistent with either the escape velocity form Earth or the escape velocity from the sun?

2. Calculate the amount of energy released at impact, assuming all the kinetic energy is "released". and the meteor has the density of a typical rocky material, or 5000 kg m^{-3}.

3. If half this energy goes into warming the atmosphere (by dissipation of shock waves, a big splash of molten material, etc.), what is the average increase in the temperature of the atmosphere? (Calculating the average increase in temperature is the same as assuming a uniform increase in temperature throughout the atmosphere).

Answer:

From Problem 1.13, the speed v is:

$$v = 17,900 \text{ m s}^{-1} \tag{1.74}$$

The escape velocity fom the Earth is

$$\left(\frac{2GM_E}{a} \right)^{1/2} = 11,000 \text{ m s}^{-1} \tag{1.75}$$

The escape velocity fom the sun, beginning at the radius of Earth's orbit R, is

$$\left(\frac{2GM_s}{R} \right)^{1/2} = 42,000 \text{ m s}^{-1} \tag{1.76}$$

These escape velocities are also the impact velocities for a meteor slamming into earth, assumed to have begun its journey with negligible kinetic energy far out in the solar system or beyond. The number in (1.75) is if only the gravity of the Earth acted on the meteor, and the number in (1.76) is if only the gravity of the sun acted. The estimated speed of the meteor is consistent with the earth intercepting an elliptical orbit of an asteroid or a comet, that has been duly accelerated by the sun.

The radius of the alleged meteor is

$$r = 3 \text{ mi} \times 1609 \frac{\text{m}}{\text{mi}} = 4.83 \text{ km} \tag{1.77}$$

The mass is

$$m = \frac{4}{3}\pi r^3 \rho = 2.36 \times 10^{15} \text{ kg} \tag{1.78}$$

The kinetic energy at impact is

$$E = M\frac{v^2}{2} = 3.8 \times 10^{23} \text{ J} \tag{1.79}$$

Suppose $\frac{1}{2}$ of this energy heats the mass of the atmosphere M,

$$\frac{1}{2}E = c_p M \Delta T \tag{1.80}$$

So

$$\Delta T = \frac{1}{2}\frac{3.8 \times 10^{23} \text{ J}}{1004 \text{ J K}^{-1} \text{ kg}^{-1} \; 5.3 \times 10^{18} \text{ kg}} = 36 \text{ K}, \tag{1.81}$$

A rather uncomfortable increase.

1.15 Melting an ice age

Between 18,000 B.P. and 6,000 B.P. the sea level rose 100 meters due to the melting of ice sheets that were on land. What fraction of the solar heat flux coming into the atmosphere was required to melt this ice during this time? Also estimate the fraction of solar radiation that was reflected by the ice during that time.

Answer:

With 71% of the Earth covered by oceans, the mass of ice melted is

$$m = .71A \times 100 \text{ m} \times 1000 \text{ kg m}^{-3} = 3.62 \times 10^{19} \text{ kg} \tag{1.82}$$

where A is the area of the Earth. The amount of heat needed to melt this ice is

$$Q = mL_f = 1.2 \times 10^{25} \text{ J} \; . \tag{1.83}$$

12,000 years is 3.79×10^{11} s. The power used to melt the ice is therefore 3.19×10^{13} W. This can be contrasted with

$$S_o \pi a^2 = 1.74 \times 10^{17} \text{ W} \quad . \tag{1.84}$$

Therefore 0.02% of the incoming radiation was being used to melt the ice. But the ice was still keeping the earth cool by its high reflectivity or albedo. If you have ever seen maps of the extensive ice sheets of an ice age, if seems reasonable that the ice could have been reflected at least 1% of the incoming radiation, or 54 times the amount of radiation used for melting.

1.16 Nonuniform insulation and a law of diminishing returns

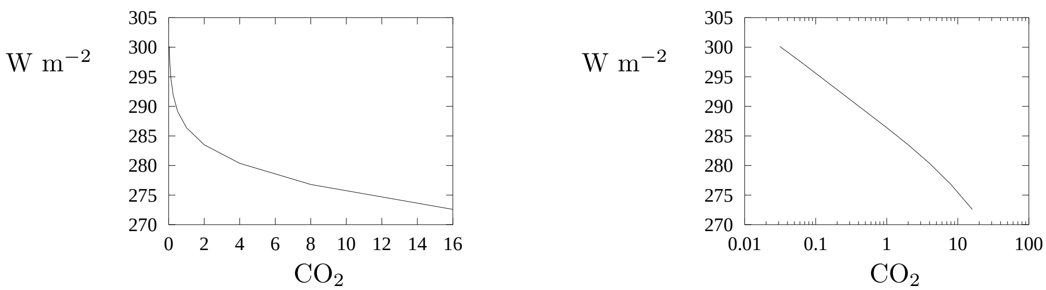

Figure 1.6: Upwards infrared flux density from the top of the atmosphere versus CO_2 concentration. The flux density is that given by NCAR's Community Radiation Model with the clear-sky mid-latitude summer profile for temperature. The CO_2 concentration is relative to the "current" concentration of 355 ppmv.

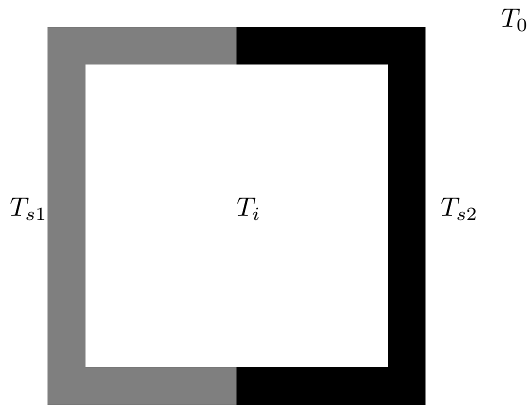

Figure 1.7: A hot box with two sorts of insulating material.

Assuming the temperature and cloud cover of Earth did not change, how would the power radiated by Earth behave as the concentration of CO_2 doubles, quadruples and so on? In Problem 1.11 we saw how the reduction in the outwards radiation to space was proportional to increases in atmospheric emissivity ϵ. To answer our question, we might think that we simply need to know how atmospheric emmissivity depends on CO_2 concentration. But actually the answer also depends on the temperature structure of the atmosphere, which is not isothermal. The calculation is rather complicated and requires the use of a computer program. An example of the output of such a program is shown in Figure 1.6. The details of the calculation are beyond the scope of these exercises. However, there are certain features of the calculation that deserve some exploration. We see that the second lot of 300 ppmv of CO_2 that we may soon acquire will provide much less of an increase in emissivity, and much less of a reduction in outwards thermal radiation, than the first lot did, as is seen in Figure 1.6. This is a rather fortunate result. Increasing the greenhouse effect with CO_2 is ultimately limited by a "law of diminishing returns", and Earth's current greenhouse is well within that regime. CO_2 absorbs predominantly in the 15 μm band, for wavelengths far outside of that band CO_2 does not have much of an effect. A crude explanation for this "law of diminishing returns" is that the wavelengths that can be absorbed well by CO_2 already are absorbed well by the current concentration of CO_2. Adding more CO_2 will not increase the emissivity as much as, say, adding a gas that absorbs in part of the spectrum where absorption is currently weak.

A similar "law of diminishing returns" can be demonstrated in analysis of a strategy to insulate a house. A box (house) is maintained at a constant temperature with a heater with power P. Two sorts of insulation are applied to the surface of the box. Over an area

A_1 there is a material with conductivity K_1 and thickness h_1. Over an area A_2 there is a material with conductivity K_2 and thickness h_2. Although the entire surface of the house cannot really be exposed to the same cross wind, we approximate the heat flux from the surface of the box into the atmosphere by the same linear relationship for the entire surface: the heat flux density is $c(T_s - T_o)$ where c is a constant that accommodates both radiative and turbulent transfer of heat from the surface. In steady state, the sum of the heat transfer by the entire surface is the power P:

$$A_1 c(T_{s1} - T_o) + A_2 c(T_{s2} - T_o) = P \tag{1.85}$$

Likewise the net heat flux conducted through the insulation is also P:

$$A_1 \frac{K_1}{h_1}(T_i - T_{s1}) + A_2 \frac{K_2}{h_2}(T_i - T_{s2}) = P \tag{1.86}$$

Also the heat flux density through any individual material must be the same as the heat flux density from the surface into the atmosphere.

$$\frac{K_1}{h_1}(T_i - T_{s1}) = c(T_{s1} - T_o) \tag{1.87}$$

and

$$\frac{K_2}{h_2}(T_i - T_{s2}) = c(T_{s2} - T_o) \tag{1.88}$$

For convenience in the algebraic manipulations, we define $k_1 \equiv \frac{K_1}{h_1}$ and $k_2 \equiv \frac{K_2}{h_2}$. Find the the power required to maintain a constant temperature T_i as a function of T_o, A_1, A_2, k_1, k_2 and c. Now, from "common sense" we expect that, if we try to save heating costs for our house, simply adding more and more insulation to the roof, without concern for insulating the walls and windows, would result in a "law of diminishing returns" as the roof becomes perfectly insulating. Show that your solution for P exhibits this law of diminishing returns.

Answer:

The four equations (1.85)-(1.88) are not independent. For example, it is simple to derive (1.85) from (1.86) with substitution from (1.87) and (1.88). Our task of solving for P in terms of the internal and external temperatures, and the physical parameters of the box, is achieved by eliminating T_{s1} and T_{s2} from (1.85) by the use of (1.87) and (1.88). The answer is

$$P = A_1 c k_1 \frac{T_i - T_o}{c + k_1} + A_2 c k_2 \frac{T_i - T_o}{c + k_2} \tag{1.89}$$

or

$$P = A_1 c K_1 \frac{T_i - T_o}{cH_1 + K_1} + A_2 c K_2 \frac{T_i - T_o}{cH_2 + K_2} \ . \tag{1.90}$$

A simple way to demonstrate a "law of diminishing returns" we simply consider $H_2 \to \infty$ and note that the asymptote for P:

$$P = A_1 c K_1 \frac{T_i - T_o}{cH_1 + K_1} \tag{1.91}$$

Obviously, at this point, it makes more sense to increase H_1 in any further attempts to reduce P.

1.17 The Gulf Stream

Suppose the Gulf Stream has a northward velocity of $v = 2$ m s^{-1}, a depth of 300 m, and a width of 100 km. The Gulf Stream has a temperature about 10 K greater, on average, than the water in the counter-currents that are flowing southward at the same latitude. How many petaWatts of energy are being transported northward by the Gulf Stream? Suppose this power is distributed over a patch of land covering parts of the north Atlantic and Europe. By this we mean the energy is radiated away to space from this patch. The patch of land covers $\frac{1}{12}$ of the Earth's surface. Suppose, in the absence of the Gulf Stream, the radiative temperature for this patch would be $T_e = 245$ K. With the energy from the Gulf Stream, what would be the increase in T_e?

Answer:

The volume of water being transported by the Gulf Stream, or the so-called *volume flux* is the cross-sectional area of the Gulf Stream times the velocity through that area:

$$F_{V,\mathrm{Gulf}} = v \times A = 2 \text{ m s}^{-1} \times 300 \text{ m} \times 10^5 \text{ m} = 6 \times 10^7 \text{ m}^3 \text{ s}^{-1} \tag{1.92}$$

The northward flux of water mass by the Gulf Stream is simply the flux of water volume times the density of water:

$$F_{M,\mathrm{Gulf}} = \rho_w \times F_{V,\mathrm{Gulf}} = 10^3 \text{ kg m}^{-3} \times 6 \times 10^7 \text{ m}^3 \text{ s}^{-1} = 6 \times 10^{10} \text{ kg s}^{-1} \tag{1.93}$$

The mass flux of the return flow is assumed to be opposite to that for the Gulf Stream:

$$F_{M,\mathrm{Return}} = -F_{M,\mathrm{Gulf}} \tag{1.94}$$

The northward flux of sensible heat being advected by one of these streams of mass is the mass flux times the specific heat times the temperature. The net heat northward flux is:

$$\begin{aligned} F_h &= F_{M,\text{Gulf}}\ c_w\ T_{\text{Gulf}} + F_{M,\text{Return}}\ c_w\ T_{\text{Return}} && (1.95) \\ &= F_{M,\text{Gulf}}\ c_w\ (T_{\text{Gulf}} - T_{\text{Return}}) && (1.96) \\ &= 6\times 10^{10}\ \text{kg s}^{-1} \times 4218\ \text{J K}^{-1}\ \text{kg}^{-1} \times 10\ \text{K} && (1.97) \\ &= 2.5\times 10^{15}\ \text{J s}^{-1} = 2.5\ \text{PW} && (1.98) \end{aligned}$$

If the heating is distributed over $\frac{1}{12}4\pi a^2$, it is equivalent to a flux density of 59 W m^{-2}, which must be radiated to space. In the absence of the Gulf Stream, $T_e = 245$ K, and the outwards long-wave radiation is $\sigma T_e^4 = 204$ W m^{-2}. With the Gulf Stream, it would be 263 W m^{-2}, or $T_e = 259$ K, which is 14 K greater. Presumably surface temperatures rise by a similar amount.

1.18 Latent heat release in the polar regions

The area A of the Earth northward of 60° N is 3.4×10^{13} m^2. Suppose northward of 60° N, the average precipitation is about 50 cm yr^{-1}. How may petaWatts of latent heat are being released into the atmosphere? (1 PW $\equiv 10^{15}$ W). How many "Tsar Bombas" per day is this equivalent to? Assume half of the water vapor for this precipitation is being advected in across the 60° N latitude circle. On average, how many W m^{-2} are thus being contributed by this advection to the energy budget of the area north of 60° N ?

Answer:

Let $\frac{\Delta M}{\Delta t}$ be the average rate of mass of water being precipitated north of 60° N. We are given that

$$\begin{aligned} \frac{\Delta M}{\Delta t} &= \rho_w \times A \times 0.5\ \text{m yr}^{-1} && (1.99) \\ &= 1000\ \text{kg m}^{-3} \times 3.4\times 10^{13}\ \text{m}^2 \times 0.5\ \text{m yr}^{-1} && (1.100) \\ &= 1.7\times 10^{16}\ \ \text{kg yr}^{-1} && (1.101) \end{aligned}$$

The rate of heating of the atmosphere by condensation is

$$\begin{aligned} \frac{\Delta Q}{\Delta t} &= L\frac{\Delta M}{\Delta t} && (1.102) \\ &= 2.5\times 10^6\ \text{J kg}^{-1} \times 1.7\times 10^{16}\ \ \text{kg yr}^{-1} \times \frac{1\ \text{yr}}{365\ \text{day}} \times \frac{1\ \text{day}}{86400\ \text{s}} && (1.103) \\ &= 1.34\times 10^{15}\ \text{W} && (1.104) \end{aligned}$$

Now,

$$1\ \text{PW} = 10^{15}\ \text{J s}^{-1} \times \frac{1\ \text{Tsar Bomba}}{4.2\times 10^{17}\ \ \text{J}} \times \frac{86400\ \text{s}}{1\ \text{day}} = 206\frac{\text{Tsar Bombas}}{\text{day}} \qquad (1.105)$$

The rate of heating by condensation is thus:

$$1.34 \text{ PW} = 276 \frac{\text{Tsar Bombas}}{\text{day}} \tag{1.106}$$

If half this energy production is being contributed from water having been evaporated from *outside* the polar region, the contribution to the polar energy budget is:

$$\frac{1}{2} \frac{\Delta Q}{\Delta t} \frac{1}{A} = 19.7 \text{ W m}^{-2} \tag{1.107}$$

1.19 Response to double CO_2

It is widely quoted that if CO_2 were to double instantaneously, and nothing else about the atmosphere changed, the net long-wave radiation out of the top of the atmosphere would decrease by 4 W m^{-2}. In fact, so much confidence exists in this number, that it is usually quoted as 4.2 W m^{-2}. (Note that in Figure 1.6, which is for a single clear-sky column, the reduction is 3 W m^{-2}). A physical explanation for this reduction is that the radiation out of the atmosphere in the CO_2 bands would be emitted, on average, from higher and cooler levels in the atmosphere, and therefore at a lesser rate.

In the net radiative flux was zero before the doubling of CO_2, after the doubling there would be a net inward flux of 4 W m^{-2} at the top of the atmosphere. The Earth would begin to warm. Forecasting the magnitude of the warming as a function of time, or calculating a new equilbrium temperature that would restore conditions of zero net flux, both require thoughtful consideration of the physics of the atmosphere and some calculus. Here we make some naive calculations about the warming that results from the flux imbalance. The calculations should help us identify important processes that would be significant in a more realistic prediction of global warming.

Suppose this radiation imbalance persisted for 50 years.

1. Suppose only the atmosphere changes temperature. What would the increase in temperature be? This calculated increase is enormous, and is unrealistic because the 4 W m^{-2} imbalance persists for 50 years, despite the fact that the atmosphere has warmed.

2. Suppose the solid earth does not change temperature, but the oceans and atmosphere uniformly increase in temperature. What would the increase in temperature be?

3. Suppose that the solid earth, oceans, and atmosphere uniformly increase in temperature. What would the increase in temperature be? Assume the specific heat for the solid earth is that listed for "soil inorganic material", or $c_e = 733$ J kg^{-1} K^{-1}.

Answer:

A net heating of 4 W m^{-2} into the top of the atmosphere for 50 years would contribute $\Delta Q = 3.22 \times 10^{24}$ J. If all this heat went into the mass of the atmosphere m_a:

$$\Delta T = \frac{\Delta Q}{c_p m_a} = 624 \text{ K} \tag{1.108}$$

Needless to say, the thermal inertia of the atmosphere alone is not going to save us from global warming. Let m_o be the mass of the ocean and m_e be the mass of (solid earth). If the entire ocean and atmosphere warm uniformly,

$$\Delta T = \frac{\Delta Q}{m_a c_p + m_o c_w} = 0.54 \text{ K} \tag{1.109}$$

This is a bit more interesting. If the oceans are mixing heat well as we experience global warming, then our calculation indicates that the thermal inertia of the oceans will delay and ameliorate the warming.

If the entire solid earth, ocean and atmosphere warm uniformly:

$$\Delta T = \frac{\Delta Q}{m_a c_p + m_o c_w + m_e c_e} = 0.0007 \text{ K} \tag{1.110}$$

If this is the case, then future generations need not fear the warming caused by anthropogenic CO_2. Unfortunately, it is not the case. Although a warming of the atmosphere and oceans would eventually warm the entire solid earth to its core, the warming would happen first in only a thin skin of solid earth.

1.20 Equilibrium with a double CO_2

As shown in Problem 1.19, a prediction of the global warming due to the instantaneous addition of CO_2 requires a model for how radiative equilibrium is restored, rather than a model that assume the radiative imbalance perpetuates. The warming would not proceed past the point where the outward radiation increases by 4 W m^{-2} due to the rise in temperature, compensating for the decrease of 4 W m^{-2} due to the increase of CO_2.

Equation (1.53) provides a simple way to estimate the rise in the surface temperature. Equation (1.53) is of form

$$\frac{S_0(1-\alpha_p)}{4} = R\sigma T_s^4 \tag{1.111}$$

For the highly idealized atmosphere of Planet X, R depended on a single parameter, the long-wave emissivity ϵ of the "slab" atmosphere: $R = 1 - \epsilon/2$. For planet Earth, R will be more complicated, and perhaps it can be known only empirically, rather than theoretically.

Applied to Earth, a value of $R = .6133$, is consistent with the observation of $T_s = 288K$. Certainly R decreases with increasing concentration of CO_2. The decrease in R for a doubling of CO_2 can be calculated given the quoted theoretical effect of doubling CO_2: a decrease in outwards radiation of 4 W m^{-2}. Using the simple conceptual model provided by (1.111), estimate the rise in T_s needed to restore equilibrium after the decrease in R.

Answer:

We are given in this problem that a doubling of CO_2 reduces R to the extent that the outward longwave radiation decreases by 4 W m^{-2}. Therefore, for a doubling of CO_2,

$$\Delta R \sigma T_s^4 = -4 \text{ W m}^{-2} \tag{1.112}$$

With $T_s = 288$ K, $\Delta R = -0.010$. Equilibrium is restored to a new value of surface temperature T_{new} if

$$R T_s^4 = (R + \Delta R) T_{new}^4 \tag{1.113}$$

This gives $T_{new} = 289.2$ K, a rise of 1.2 K. The more detailed global climate models, often quoted in the news, commonly predict an increase of between 2 K and 4 K, because of *positive feedbacks* in the warming process. See Book 2.

1.21 Energy ration

Suppose the 1990 concentration of CO_2 doubles by year 2040. Suppose the doubling is entirely due to burning methane, the CO_2 product staying entirely in the atmosphere, and increasing at a constant rate. What fraction of total solar radiation absorbed by Earth is equal to this energy production rate? Hint: you should recall that net solar power absorbed by Earth is $S_o(1 - \alpha_p)\pi a^2$. How many Watts of energy per person are being produced? To make your math easy, assume the population of Earth is a constant 8 billion people. We choose to assume methane is our only fossil fuel because our chemistry books give us the following precise, and easy to use, fact:

CH_4 (g) + 2 O_2 (g) $\rightarrow$ CO_2 (g) + 2 H_2O(l), ΔH = -890.3 kJ

A table of "Fuel and Energy Source Codes and Emission Coefficients" states methane produces 115.26 pounds of CO_2 per million BTU. Verify that this is consistent with the text book formula. 1 BTU = 1055 J. 1 kg = 2.2 pound. The same table states that typical coal produces 212.7 pounds of CO_2 per million BTU. Coal-fired electricity generation is not perfectly efficient, and is alleged to produce 1.5 pounds of CO_2 per kilowatt-hour. How does your answer change if all of our energy needs are met by coal-fired electricity?

Answer:

The molecular mass of CO_2 is 44.01 g mol^{-1}. The energy released by burning methane is

$$890.3 \text{ kJ mol}^{-1} \div 44.01 \text{ g mol}^{-1} = 2.02 \times 10^7 \ \text{ J kg}^{-1} \tag{1.114}$$

Note the "kg" here refers to the kilograms of CO_2 produced. A doubling of atmospheric CO_2 means an addition of 2.75×10^{15} kg of CO_2, and 5.56×10^{22} J of energy could be produced by burning methane alone. Distributed over 50 years, this is 3.53×10^{13} W. With $S_o(1 - \alpha_p)\pi a^2 = 1.22 \times 10^{17}$ W, the energy produced by burning methane is 0.03% of the incoming solar energy.

Distributed over 8 billion people, 5.36×10^{13} W, allows for 4400 W per person. This is the individual "ration", assuming no attempts to prevent a doubling of CO_2 by 2050. My monitor and computer consume about 10% of my personal ration. With my use of an automobile and central heat and air, I undoubtedly far exceed it. (Not to mention the street lights, food productions, etc. that I require.)

The picture is a bit more bleak when we consider more realistic estimates. With burning coal we obtain 1.1×10^7 J kg^{-1}. With that number, the ration is approximately $\frac{1}{2}$ the original estimate. Coal-fired electricity generation produces 5.3×10^6 J kg^{-1}, leaving us with a mere 1150 W per person!

About half of the CO_2 being released into the atmosphere is not accumulating there. The reason is not completely understood; perhaps it is being absorbed by the oceans or plants. If half of this sequestering of CO_2 continues, then our energy ration doubles to 2300 W per person.

1.22 Nuclear energy

Several decades ago the primary public debate about energy concerned nuclear energy, rather than the use of fossil fuels. Besides the concern of safety, we would occasionally hear concerns that nuclear energy would cause global warming simply because it would be such a cheap, prolific source of energy - and all the energy would ultimately be converted to heat that would need to be radiated away to space.

In Problem 1.21, we calculated an energy ration of 2300 W per person of fossil fuel burning would double CO_2 in 50 years. In Problem 1.19 we learned that a doubling of CO_2, without a temperature adjustment, would reduce the outward radiation by 4 W m^{-2}. A temperature increase of about 1.2 K would be required to restore equilibrium, based on a simple model without any feedbacks.

Here we consider the effect of the waste heat of energy production warming the planet. Suppose our entire energy ration was supplied by nuclear energy. The energy balance for Earth could be written as:

$$S_o(1-\alpha_p)\pi a^2 + P = 4\pi a^2 \sigma T_e^4 \tag{1.115}$$

where P is the power of energy production. Compare T_e with $P = 0$ and P for an energy *consumption* ration of 2300 W per person. Assume all energy is supplied by nuclear power plants, and that the waste heat before the energy is delivered to the consumer is twice the amount consumed. Therefore the energy *production* ration is 6900 W per person.

Answer:

As in Problem 1.21, $S_o(1-\alpha_p)\pi a^2 = 1.22 \times 10^{17}$ W. With $P = 0$, we calculate $T_e =$ 254.8713 K. For 8 billion people, $P = 5.52 \times 10^{13}$, and T_e is increased by 0.028 K, which is much less than the warming caused by enhancing the greenhouse effect.

Appendix A

Useful Numerical Values

Fundamental constants	
Universal gas constant (R^*)	8.3143 J K^{-1} mol^{-1}
Boltzmann's constant (k)	1.38 $\times 10^{-23}$ J K^{-1}
Stefan-Boltzmann constant (σ)	5.67 $\times 10^{-8}$ W m^{-2} K^{-4}
Planck's constant ($\hbar$)	6.63 $\times 10^{-34}$ J s
Speed of light (c^*)	2.998 $\times 10^{8}$ m s^{-1}
Gravitational constant	6.67 $\times 10^{-11}$ Nt m^2 kg^{-2}
Avogadro Number (N_A)	6.022 $\times 10^{23}$ particles mol^{-1}
Sun	
Luminosity	3.92 $\times 10^{26}$ W
Mass	1.99 $\times 10^{30}$ kg
Radius	6.96 $\times 10^{8}$ m
Earth	
Average radius (a)	6.37 $\times 10^{6}$ m
Equitorial radius	6.378 $\times 10^{6}$ m
Polar radius	6.357 $\times 10^{6}$ m
Standard gravity	9.80665 m s^{-2}
Mass of Earth	5.983 $\times 10^{24}$ kg
Mass of ocean	1.4 $\times 10^{21}$ kg
Mass of atmosphere	5.136 $\times 10^{18}$ kg

Mean angular rotation rate (Ω)	7.292×10^{-5} rad s^{-1}
Solar constant (S_o)	1367 ± 2 W m^{-2}
Mean distance from sun	1.496×10^{11} m
Planetary albedo	0.30
Dry air	
Average molecular weight(m_a)	28.97 g mol^{-1}
Gas constant (R)	287 J K^{-1} kg^{-1}
Density as 0° C and 101,325 Pa	1.293 kg m^{-3}
Specific heat at constant pressure (c_p)	1004 J K^{-1}kg^{-1}
Specific heat at constant volume (c_v)	717 J K^{-1}kg^{-1}
Water	
Molecular weight (m_v)	18.016 g mol^{-1}
Gas constant for vapor ($R_v = R^*/m_w$)	461 J K^{-1} kg^{-1}
Density of pure water at 0° C	1000 kg m^{-3}
Density of ice at 0° C	917 kg m^{-3}
Specific heat of vapor at constant pressure	1952 J K^{-1} kg^{-1}
Specific heat of vapor at constant volume	1463 J K^{-1} kg^{-1}
Specific heat of liquid water at 0° C	4218 J K^{-1} kg^{-1}
Specific heat of ice at 0° C	2106 J K^{-1} kg^{-1}
Latent heat of vaporization at 0° C	2.5×10^{6} J kg^{-1}
Latent heat of vaporization at 100° C	2.25×10^{6} J kg^{-1}
Latent heat of fusion at 0° C	3.34×10^{5} J kg^{-1}
CO_2	
Molecular weight (m_v)	44.01 g mol^{-1}
Density of dry ice	1500 kg m^{-3}
parts per million by volume in dry air (1990)	353 ppmv
Mass in atmosphere (1990)	2.76×10^{15} kg
Soil	
Specific heat of inorganic soil	733 J kg^{-1} K^{-1}
Density of inorganic soil	2600 kg m^{-3}

Specific heat of organic soil	1921 J kg^{-1} K^{-1}
Density of organic soil	1300 kg m^{-3}
Thermal conductivity of dry peat	0.1 W m^{-1} K^{-1}
Thermal conductivity of typical soil	1.0 W m^{-1} K^{-1}
Thermal conductivity of wet sand	2.5 W m^{-1} K^{-1}