Well-Tempered Climate Theory

Book 2

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Abstract

J. S. Bach's *Well-tempered Clavier* consists of 48 Preludes and Fugues. My collection of approximately 48 exercises in the theory of climate needed a title. Here is what Frédéric Chopin had to say about J. S. Bach's *Well-tempered Clavier*: "Play Bach's Preludes and Fugues every day... This is the best school; no one will ever create a better... Without Bach, you cannot have freedom in the fingers, nor a clear or beautiful tone... Everything he does is perfect; it is not possible to imagine it otherwise, and the slightest change would spoil everything."

Thus, the relation of these exercises with the masterpieces of Bach is tenuous indeed. I forsee many changes in these exercises as typos and errors are discovered, and inadequate explanations induce complaints from the students. Also, the typesetting still suffers from inconsistencies. But the intent is that these exercises enjoy wide practice by those seeking to understand and participate in the debate about global warming.

The exercises offered here are distilled for the point of highlighting an important physical concept. They do not provide an authoritative calculation of a number that can be used in a policy decision. In fact, the (approximately) 24 exercises of Book 1 are so distilled that they are devoid of calculus or trigonometry. Students of Bach never had it so easy. Book 2 will offer no such reprieve. Book 2 currently offers only 4 exercises. Needless to say, it is a work in progress.

These exercises are, or course, not a textbook. It is expected that students assigned these exercise will need plenty of supplementary material.

A Table of Useful Numerical Values appears at the end of these exercises. I am indebted to *Global Physical Climatology* by Dennis Hartmann for most of the values.

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Book 2

Climate theory with calculus

2.1 Spherical trigonometry

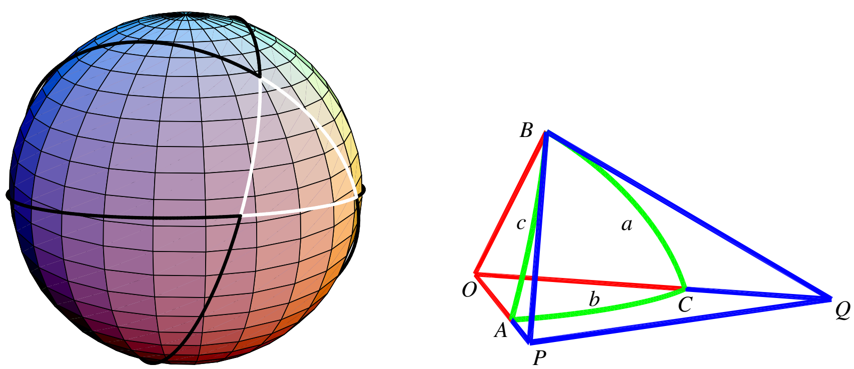

Figure 2.1: A great circle is formed by the intersection of a plane through the center of a sphere and the surface of the sphere. Three arcs of great circles can form a spherical triangle. The points at the vertices of the spherical triangle are labelled A, B and C. The angles at those points have measure A, B and C. The sides opposite A, B and C have arc angle a, b and c. The origin, or center of the sphere, is labelled O. Point P is at the intersection of a line through O and A and a plane tangent to the sphere at B. A similar definition applies to Q.

Note in Figure (2.1) that A is measured in a plane tangent to the sphere, while a is the arc angle measured in a plane through the center of the sphere. $A \neq a$, but is related by:

$$\cos a = \cos c \ \cos b \ + \ \sin c \ \sin b \ \cos A. \tag{2.1}$$

which is known as the law of cosines for a spherical triangle. The calculations for the distribution of solar radiation over a sphere will be based on this law. Our first task before we use (2.1) is to prove it; two ways are offered here.

1. The first proof follows from noticing that since angle A is the angle of intersection between planes OAB and OAC, the value of this angle will be the same as the angle between the normal vectors to those two planes. The normal vectors to those planes are easily defined by taking a cross product between appropriate pairs of unit vectors defined $\overrightarrow{OA}/|\overrightarrow{OA}| \equiv \hat{a}$, $\overrightarrow{OB}/|\overrightarrow{OB}| \equiv \hat{b}$ and $\overrightarrow{OC}/|\overrightarrow{OC}| \equiv \hat{c}$.

2. The second method involves using the law of cosines to the plane triangles OPQ and BPQ and the Pythagorean theorem to the right triangles OBP and OBQ. Actually, what is proven is that

$$\cos b = \cos c \ \cos a \ + \ \sin c \ \sin a \ \cos B. \tag{2.2}$$

   and (2.1) follows by relabeling the angles.

Answer:

Here are some details in the two proofs:

1. The magnitude of $\hat{a} \times \hat{b}$ is $\sin c$ and the magnitude of $\hat{a} \times \hat{c}$ is $\sin b$. The two vectors are the normals to the planes that intersect at an angle A. The angle between the normals is the same as the angle of intersection between the planes. The cosine of the angle between the planes, and the normals, is $\cos A$. Therefore, using the familiar geometric definition of a dot product:

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{c}) = \sin c \ \sin b \ \cos A \tag{2.3}$$

   Next using

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{c}) = \hat{b} \cdot \hat{c} \ \hat{a} \cdot \hat{a} - \hat{b} \cdot \hat{a} \ \hat{a} \cdot \hat{c} \tag{2.4}$$

   it is shown by finding the angle between the pairs of unit vectors that

$$(\hat{a} \times \hat{b}) \cdot (\hat{a} \times \hat{c}) = \cos a - \cos c \cos b \tag{2.5}$$

   and thus (2.1) follows.

2. Here is the alternative proof. Applying the law cosines for plane triangles OPQ and BPQ we have:

$$(PQ)^2 = (OQ)^2 + (OP)^2 - 2(OQ)(OP)\cos b \tag{2.6}$$

and

$$(PQ)^2 = (BQ)^2 + (BP)^2 - 2(BQ)(BP)\cos B \tag{2.7}$$

Taking the difference of these equations gives

$$0 = (OQ)^2 - (BQ)^2 + (OP)^2 - (BP)^2 - 2(OQ)(OP)\cos b + 2(BQ)(BP)\cos B \tag{2.8}$$

With the Pythagorean theorem giving

$$(OQ)^2 = (BQ)^2 + (OB)^2 \tag{2.9}$$

$$(OP)^2 = (BP)^2 + (OB)^2 \tag{2.10}$$

$$\tag{2.11}$$

$$0 = 2(OB)^2 - 2(OQ)(OP)\cos b + 2(BQ)(BP)\cos B \tag{2.12}$$

and thus

$$\cos b = \frac{OB}{OQ} \cdot \frac{OB}{OP} + \frac{BQ}{OQ} \cdot \frac{BP}{OP} \cos B \tag{2.13}$$

and (2.2) follows.

2.2 Solar radiation on a sphere

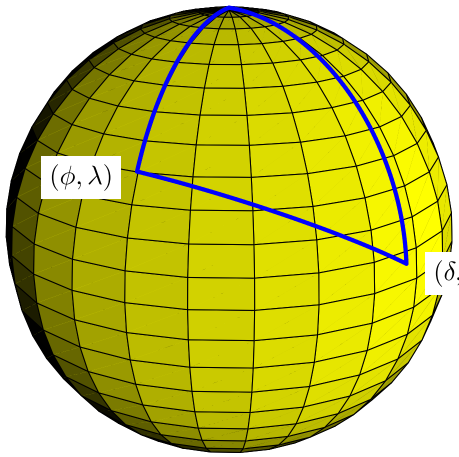

Figure 2.2: The law of cosines for a spherical triangle can be used to calculate the zenith angle θ_s at a lat-lon point (ϕ, λ). The subsolar point is at latitude δ (which is also called the *solar declination angle*) and longitude $\lambda + h$. Here the *hour angle* h is the relative longitude of the subsolar point from the point in question.

In order to apply (2.1) to the situation in figure 2.2, let A be the north pole, let B be the subsolar point at latitude δ and let C be a point at (ϕ, λ) where we wish to know the solar zenith angle θ_s. By definition, the subsolar point is the unique location on the globe where $\theta_s = 0$. In (2.1), the angle $A = h$.

1. Why is $\theta_s = a$?

2. Show using (2.1), the solar zenith angle θ_s is related to latitude ϕ, declination angle δ and hour angle h by:

$$\cos\theta_s = \sin\phi\sin\delta + \cos\phi\cos\delta\cos h \tag{2.14}$$

If the distance of Earth from the sun is R_E, and if the mean distance is R_o, the solar flux density into a plane tangent to the sphere, but at "the top" of the atmosphere:

$$Q = S_o\frac{R_o^2}{R_E^2}\cos\theta_s \text{ when } \cos\theta_s > 0 \tag{2.15}$$

$$Q = 0 \text{ when } \cos\theta_s \leq 0. \tag{2.16}$$

Sunrise occurs at hour angle h_o, (and sunset occurs at hour angle $-h_o$), given by

$$\sin\phi\sin\delta + \cos\phi\cos\delta\cos h_o = 0. \tag{2.17}$$

Show that

$$\overline{Q}^{day} = \frac{S_o}{\pi}\frac{R_o^2}{R_E^2}\left(h_o\sin\phi\sin\delta + \cos\phi\cos\delta\sin h_o\right). \tag{2.18}$$

Answer:

The average value of Q over a day is:

$$\overline{Q}^{day} = \frac{\int_{h=\pi}^{h=-\pi} Q dh}{\int_{h=\pi}^{h=-\pi} dh}. \tag{2.19}$$

Now $\int_{h=\pi}^{h=-\pi} dh = -2\pi$ and, if $\frac{R_o^2}{R_E^2}$ is nearly constant over the course of a day,

$$\begin{aligned}
\int_{h=\pi}^{h=-\pi} Q dh &= \int_{h=h_o}^{h=-h_o} Q dh = S_o\frac{R_o^2}{R_E^2}\int_{h=h_o}^{h=-h_o}\cos\theta_s dh \\
&= S_o\frac{R_o^2}{R_E^2}\left[h\sin\phi\sin\delta + \cos\phi\cos\delta\sin h\right]_{h=h_o}^{h=-h_o}. \\
&= -2S_o\frac{R_o^2}{R_E^2}\left[h_o\sin\phi\sin\delta + \cos\phi\cos\delta\sin h\right] \qquad (2.20)
\end{aligned}$$

2.3 Seasonal variation of solar radiation at the top of the atmosphere

Over the course of year, the declination angle δ oscillates within the range of the Earth's obliquity, which is the angle the rotation axis makes with the normal vector to the orbital plane. The current value of obliquity is 23.45°.

Make a plot of $\overline{Q}^{day}$ as a function of latitude, when $\delta = 23.45°$ (N.H. summer), $\delta = 0°$ (equinox) and $\delta = -23.45°$ (S.H. winter). For this exercise, assume the orbit of the Earth is exactly circular, with $R_E = R_o$. (This approximation introduces some errors). **Answer:**

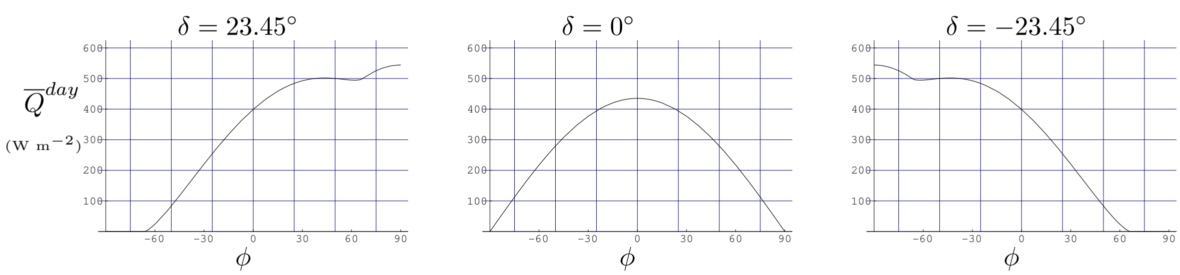

Figure 2.3: Daily averaged radiation $\overline{Q}^{day}$ at the top of the atmosphere as a function of latitude ϕ, for various δ.

2.4 Summer radiation at high latitudes and the ice ages

In the elliptical orbit of Earth, $R_E = R_o(1-\epsilon^2)/(1+\epsilon\cos\nu)$ where ν is the angle of the orbital path around the sun, measured from perihelion, and ϵ is the eccentricity of the elliptical orbit. Calculate the $\overline{Q}^{day}$ at 50 N on the day of the summer solstice with the following parameters:

1. Hypothetical circular conditions: $\delta = 23.45°$ and $\epsilon = 0.0$. (499 W m^{-2})
2. Current conditions: $\delta = 23.45°$, $\epsilon = .015$ and summer solstice is (nearly) the time of aphelion. (484 W m^{-2})
3. 10,000 years ago: $\delta = 24.2°$, $\epsilon = .016$ and summer solstice is the time of perihelion. (523 W m^{-2})
4. 115,000 years ago: $\delta = 22.3°$, $\epsilon = .042$ and summer solstice is the time of aphelion. (450 W m^{-2})
5. 125,000 years ago: $\delta = 24.0°$, $\epsilon = .042$ and summer solstice is the time of perihelion. (551 W m^{-2})

What do the above calculations imply about the cause of ice ages, if 125,000 years ago that last interglacial was at its peak, 115,000 year ago a new ice age was beginning, 10,000 years ago the last ice age was coming to an end?

Appendix A

Useful Numerical Values

Fundamental constants

Universal gas constant (R^*)	8.3143 J K^{-1} mol^{-1}
Boltzmann's constant (k)	1.38 $\times 10^{-23}$ J K^{-1}
Stefan-Boltzmann constant (σ)	5.67 $\times 19^{-8}$ W m^{-2} K^{-4}
Planck's constant ($\hbar$)	6.63 $\times 10^{-34}$ J s
Speed of light (c^*)	2.998 $\times 10^{8}$ m s^{-1}
Gravitational constant	6.67 $\times 10^{-11}$ Nt m^2 kg^{-2}
Avogadro Number (N_A)	6.022 $\times 10^{23}$ particles mol^{-1}

Sun

Luminosity	3.92 $\times 10^{26}$ W
Mass	1.99 $\times 10^{30}$ kg
Radius	6.96 $\times 10^{8}$ m

Earth

Average radius (a)	6.37 $\times 10^{6}$ m
Equitorial radius	6.378 $\times 10^{6}$ m
Polar radius	6.357 $\times 10^{6}$ m
Standard gravity	9.80665 m s^{-2}
Mass of Earth	5.983 $\times 10^{24}$ kg
Mass of ocean	1.4 $\times 10^{21}$ kg
Mass of atmosphere	5.136 $\times 10^{18}$ kg

Mean angular rotation rate (Ω)	7.292×10^{-5} rad s^{-1}
Solar constant (S_o)	1367 ± 2 W m^{-2}
Mean distance from sun	1.496×10^{11} m
Planetary albedo	0.30
Dry air	
Average molecular weight(m_a)	28.97 g mol^{-1}
Gas constant (R)	287 J K^{-1} kg^{-1}
Density as 0° C and 101,325 Pa	1.293 kg m^{-3}
Specific heat at constant pressure (c_p)	1004 J $K^{-1}kg^{-1}$
Specific heat at constant volume (c_v)	717 J $K^{-1}kg^{-1}$
Water	
Molecular weight (m_v)	18.016 g mol^{-1}
Gas constant for vapor ($R_v = R^*/m_w$)	461 J K^{-1} kg^{-1}
Density of pure water at 0° C	1000 kg m^{-3}
Density of ice at 0° C	917 kg m^{-3}
Specific heat of vapor at constant pressure	1952 J K^{-1} kg^{-1}
Specific heat of vapor at constant volume	1463 J K^{-1} kg^{-1}
Specific heat of liquid water at 0° C	4218 J K^{-1} kg^{-1}
Specific heat of ice at 0° C	2106 J K^{-1} kg^{-1}
Latent heat of vaporization at 0° C	2.5×10^{6} J kg^{-1}
Latent heat of vaporization at 100° C	2.25×10^{6} J kg^{-1}
Latent heat of fusion at 0° C	3.34×10^{5} J kg^{-1}
CO_2	
Molecular weight (m_v)	44.01 g mol^{-1}
Density of dry ice	1500 kg m^{-3}
parts per million by volume in dry air (1990)	353 ppmv
Mass in atmosphere (1990)	2.76×10^{15} kg
Soil	
Specific heat of inorganic soil	733 J kg^{-1} K^{-1}
Density of inorganic soil	2600 kg m^{-3}

Specific heat of organic soil	1921 J kg^{-1} K^{-1}
Density of organic soil	1300 kg m^{-3}
Thermal conductivity of dry peat	0.1 W m^{-1} K^{-1}
Thermal conductivity of typical soil	1.0 W m^{-1} K^{-1}
Thermal conductivity of wet sand	2.5 W m^{-1} K^{-1}